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ABSTRACT

This book is part 2 of a 2-part manual for teachers using SMSG text materials for grade 4. The purpose for each of 5 chapters is stated and mathematical background for the teacher is presented. Detailed lesson plans are then provided, including sequences of statements and questions, activities, and exercise sets with answers. Needed materials and vocabulary are listed. Chapter topics include: (1) addition and subtraction; (2) multiplication and division; (3) common geometric figures; (4) linear measurement; and (5) rational numbers. (MN)

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Mathematics for the Elementary School, Grade 4

Teacher's Commentary, Part II

REVISED EDITION

Prepared under the supervision of the
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Chapter 6

PROPERTIES AND TECHNIQUES OF ADDITION AND SUBTRACTION II

PURPOSE OF THE UNIT

1. To help children understand the techniques of adding and subtracting numbers whose numerals have many digits and their dependence on the properties of our system of numeration, the commutative property, and the associative property of addition
2. To help children understand that if they know the addition facts and the properties of addition and subtraction, they may add and subtract numbers whose numerals have many digits
3. To help children develop skill in addition and subtraction and in checking these operations
4. To help children further their problem solving ability through the use of mathematical sentences in situations requiring addition and subtraction of numbers whose numerals have many digits with particular emphasis on learning when subtraction is applicable

Children who have learned previously the technique of addition and subtraction thoroughly may move through the first half of this unit very quickly. Others should move more slowly. However, the development of the meaning basic to the techniques is important and should not be slighted with any group. Indeed, it should be given emphasis. It and the development of an ability to solve problems are the major purposes of this unit.

MATHEMATICAL BACKGROUND

The answer to the question, "What is the sum of 5 and 2?", is 5 + 2. Ordinarily the question is interpreted to mean, "What is the decimal numeral for the sum of 5 and 2?" For this question the answer is 7. Similarly, the answer to, "What is added to 4 so the sum will be 7?", **is 7 - 4. As a decimal** numeral, 7 - 4 is 3.

The answer, 892 + 367 in response to "What is the sum of 892 and 367?" is correct but is ordinarily not the most convenient one. The decimal numeral 1,259 is the **expected response. In learning the process of addition then, a primary objective is to write a sum such as 892 + 367 in the form of a decimal numeral.**

If 42 and 37 are the numbers added, the result is 7 tens and 9. This sum can be written directly as 79. However, if 67 and 58 are the numbers and the operation is addition the result is 11 tens and 15. This sum cannot be written directly as a decimal numeral but must be thought of as 1 hundred, 2 tens, and 5. Then it is written as 125.

Similarly, when the operation of subtraction is used with numbers whose numerals have more than one digit, the naming of the number, which is the sum, as a decimal numeral often requires careful thinking. If 49 and 23 are the numbers operated on, the unknown addend is 2 tens and 6. This result may be written directly as 26. If, however, 32 and 17 are the numbers operated on, the 32 must be thought of as 2 tens and 12. This makes it convenient to subtract 1 ten and 7. The result, 1 ten and 5, is now written as 15.

Thinking of 32 as 3 tens and 2 ones or as 2 tens and 12 ones has often been referred to as "renaming 32". "Renaming a number" has been used widely in some of the previous units. It is clear on a moment's reflection that a number can be renamed in an infinite number of ways. For example, the number 8, can be renamed as m - n where n may represent any number we please and $m = n + 8$. In this particular unit we are almost entirely concerned with a type of renaming which is particularly useful in addition and subtraction involving larger numbers.

This type of renaming is not identified by any word or phrase in this unit and the word rename will not be used here. Instead, terminology such as the following is used:

"523 may be thought of as 500 + 20 + 3", or

"Another name for 523 is 500 + 20 + 3", or

"523 may be expressed as 500 + 20 + 3". At the same time the idea of different names for the same number is basic for learning the processes of addition and subtraction. It must be comprehended if pupils are to compute sums and unknown addends with understanding.

Consider the following example which should make clear the need to learn how to express numbers in different forms and should indicate its useful application in addition and subtraction.

Example. Find the number that must be added to 376 so that 523 will be the sum. If written as a mathematical sentence it may take the form $376 + n = 523$. If the pupil immediately recognizes that 147 is the addend that with the given addend, 376, yields the sum, 523, there is little need of proceeding further. A response to such a simple request as, "Find the number that must be added to 6 so that 10 will be the sum," would doubtless be immediate from the simple recall of the proper addition fact. But in the example under consideration the numbers are so large that recognition will not be immediate. Hence the process will go something like this:

First, write the exercise.
$$\begin{array}{r} 523 \\ -\ \underline{376} \end{array}$$

Now the need to express 523 and 376 in forms other than as decimal numerals is apparent. The objective is to put a greater number of ones in the ones' place in the sum than in the ones' place in the addend, to put more tens in the tens' place in the sum than in the tens' place in the addend, etc.

So we may write $523 = 500 + 20 + 3$

$376 = 300 + 70 + 6$

as an initial step in accomplishing our purpose.

Next $523 = 500 + 10 + 13$

$376 = 300 + 70 + 6$

and $523 = 400 + 110 + 13$

$376 = 300 + 70 + 6$

so that $523 - 376 = 100 + 40 + 7 = 147$

There is no need for a special term to describe this procedure to ourselves or the pupils. We avoid the need for a term by saying, "We write 523 as 500 + 20 + 3" and write as indicated. The primary issue is recognizing the need for more ones in ones' place of the sum than in the ones' place of the addend, more tens in the tens' place of the sum than in the tens' place of the addend, etc.

In addition there is less difficulty. We simply name the addends to exhibit the number of ones, the number of tens and so on. Then the ones, tens, hundreds and so on are added.

For example:

$$\begin{aligned} 249 &= 200 + 40 + 9 \\ \underline{676} &\underline{= 600 + 70 + 6} \\ \text{Sum} &= 800 + 110 + 15 \\ &= (800 + 100) + (10 + 10) + 5 \\ &= 900 + 20 + 5 \\ &= 925. \end{aligned}$$

Here the numeral, $800 + 110 + 15$ is expressed as the decimal numeral by applying properties of our system of numeration and both the commutative and associative properties for addition.

It is recognized, of course, that this explanation is long and wordy in written exposition. It can, however, be made fairly brief in presentation to the pupil, and in his subsequent execution of it be made briefer. Details are supplied in an attempt to explain the basis for **"borrowing" and "carrying" and an ultimate discard of these terms which are frequently executed properly but almost universally misunderstood.**

An important property of subtraction in the symbolism of the mathematician is:

For a pair of whole numbers named in the form $(a + b)$ and $(c + d)$

$$(a + b) - (c + d) = (a - c) + (b - d).$$

In this unit it is assumed that $a + b$ is greater than $c + d$.

Here is an illustration of this property for $68 - 42$. It shows that in writing a subtraction in the vertical form the property is applied automatically

$$\begin{aligned} 68 - 42 &= (60 + 8) - (40 + 2) \\ &= (60 - 40) + (8 - 2) \\ &= 20 + 6 \\ &= 26. \end{aligned} \qquad \begin{array}{r} 68 \\ -\ \underline{42} \\ 26 \end{array}$$

The relation between the illustration and the statement of the property is seen if you think of a replaced by 60, b replaced by 8, c replaced by 40 and d replaced by 2.

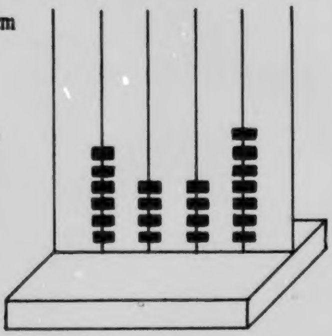
The property is applicable to other subtractions such as $68 - 49$ or $352 - 187$. More steps are required however.

TEACHING THE UNIT

DIFFERENT NAMES FOR THE SAME NUMBER

Objective: (a) To help pupils review the idea of many different names for the same number
(b) To help pupils illustrate different names for a number on an abacus

Materials Needed: An abacus with 20 beads on each of four rods



The figure at the right shows one form of an abacus. It has rods inserted in the base with beads that may be removed when not in use. Other forms of the abacus are equally useful.

The rods of the abacus correspond to places in a numeral. Moving from right to left, the first rod corresponds to ones' place, the second to tens' place, etc. The number of beads on the first rod is the number of ones, the number of beads on the second rod is the number of tens, etc. The numeral represented on the abacus here is 6 thousands, 4 hundreds, 4 tens, 7 ones or 6447.

Exploration:

|| Parts of the following material are written as if the teacher were talking to his class. The answers he wishes to elicit from pupils in response to questions are included in parentheses. Other parts of the exploratory material for this unit are in the form of suggestions to the teacher. These are also descriptions of procedures that are to be used as a basis for discovery and exploration of concepts and properties. These remarks are written between double vertical lines. ||

Some of the exploratory material is contained in the Pupils' Book. This is the case with the first section, Different Names for the Same Number. Children should soon recognize that the phrases "Different Names for the Same Number" and "Renaming a Number" have the same meaning.

The teacher may have pupils open their books to page 303 and answer and discuss the questions. After this development, Exercise Set 1 may be completed by each pupil individually.

Chapter 6

PROPERTIES AND TECHNIQUES OF ADDITION AND SUBTRACTION II

DIFFERENT NAMES FOR THE SAME NUMBER

There are many ways of naming a number. The decimal numeral for 40 + 2 is 42. It may also be named in other ways.

1. Nan says that all the names below are for the same number. Do you agree? 27 - 3; 24 + 0; 10 + 14; 25 - 1; 2 tens and 4; 1 ten and 14. (yes)

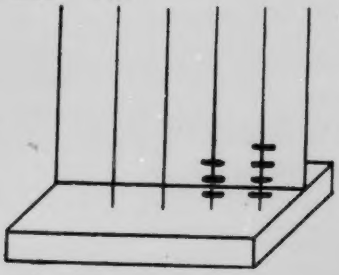
2. What is the decimal numeral for 40 + 15? (55) State five other names for 40 + 15. (answers will vary)

3. (a) Is 234 = 200 + 30 + 4? (yes) (c) Is 234 = 200 + 20 + 14? (yes)
   (b) Is 234 = 200 + 10 + 24? (yes) (d) Is 234 = 100 + 130 + 4? (yes)

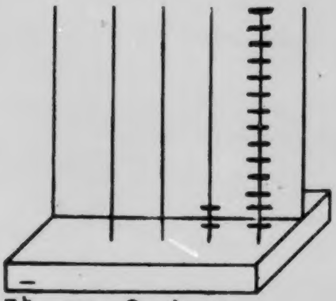
4. You may think of 67 as 6 tens and 7 ones or as 5 tens and 17 ones. What are other names for 67?

   May we think of 726
   as 700 + 20 + 6?
   as 700 + 10 + 16? (yes)
   as 600 + 120 + 6?

   Different names for a number are often shown on an abacus. How is 34 named on each abacus at the right?



34 as 3 tens and 4 ones



34 as 2 tens and 14 ones

5. Tell two different names for each of these numbers. Show each on the abacus. (Answers will vary)
   (a) 46 (b) 97 (c) 263

15

Exercise Set 1

Copy the numerals 1 - 10 on your paper. Next to each write the correct answers to complete this chart.

	Decimal Numeral	Another Name for the Number
1	(137)	120 + 17
2	(1378)	1200 + 160 + 18
3	523	(400 + 110 + 13)
4	(78)	6 tens + 18
5	96	(80 + 16)
6	(558)	4 hundreds + 15 tens + 8
7	238	(200 + 20 + 18)
8	49	(3 tens + 19)
9	(1749)	15 hundreds + 23 tens + 19
10	1,526	(14 hundreds + 12 tens + 6)

For each of exercises 11 - 13 write > or < so each mathematical sentence will be true. In exercise 11, is 1000 + 300 + 60 + 16 another name for 1376? (yes)

11. 1,378 (>) 1000 + 300 + 60 + 16

12. 2,874 (>) 1 thousand + 17 hundreds + 16 tens + 4 ones

13. 4,926 (>) 3 thousands + 18 hundreds + 11 tens + 6 ones

REVIEW OF ADDITION

Objective: To help pupils review the techniques of addition where naming the sum as a decimal numeral is a simple procedure

Exploration:

Here are four addition exercises on the chalkboard.

(a) $3 + 4$ (c) $300 + 400$

(b) $30 + 40$ (d) $3000 + 4000$

What is one numeral for the sum in each of these? (In (a) it is 7. In (b) it is 70, which means 7 tens. In (c) it is 700, which means 7 hundreds. In (d) it is 7000, which means 7 thousands.)

Exercises like (b), (c), and (d) are often written in column form because it is an easy way to group the ones, tens, hundreds, etc.

(b) $\begin{array}{r} 30 \\ + \underline{40} \end{array}$ (c) $\begin{array}{r} 300 \\ + \underline{400} \end{array}$ (d) $\begin{array}{r} 3000 \\ + \underline{4000} \end{array}$

How do you add 23 and 45? (The sum of 20 and 40 is 60. The sum of 3 and 5 is 8. The sum of 60 and 8 is 68.) This is an easy exercise but we need to study it because you have used some important properties. Let us find them.

You thought of 23 and 45 in another way. What was it? (We found other names for them. $23 = 20 + 3$. $45 = 40 + 5$.)

You thought of $23 + 45$ as $(20 + 3) + (40 + 5)$? (Yes, but we used the commutative property and associative property to get $(20 + 40) + (3 + 5)$. Why did you want this order? (So we may get $60 + 8$ or 68.)

May we write our work like this?

$$23 + 45 = (20 + 3) + (40 + 5) = (20 + 40) + (3 + 5) = 60 + 8 = 68$$

We may write our work for the pupils as in the preceding statement. But the best pupils, possibly all the pupils, should be aware that the following steps, or similar ones, are used.

$$(20+3)+(40+5) = (20+3)+(5+40)$$

Commutative Property

$$= 20+[3+(5+40)]$$

Associative Property

$$= 20+[(3+5)+40]$$

Associative Property

$$= 20+[8+40]$$

Adding 3 and 5

$$= 20+[40+8]$$

Commutative Property

$$= [20+40]+8$$

Associative Property

$$= 60+8$$

Adding 20 and 40

$$= 68$$

Adding 60 and 8.

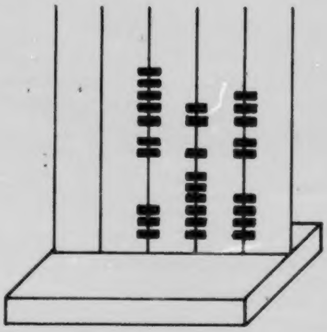
The column form of writing the addends is helpful in any exercise like this. We can write all we have said in another form.

$$\begin{array}{rl} 23 = & 20 + 3 \\ 45 = & \underline{40 + 5} \\ & 60 + 8 = 68 \end{array}$$

Let us do another exercise with more addends and show the addition on an abacus and study the properties we use.

Show the three addends for this addition:

$$\begin{array}{r} 523 \\ 212 \\ \underline{364} \end{array}$$



What is a convenient form for writing the addends? (The form shown on the right should be written on the board.)

$$\begin{aligned} 523 &= 500 + 20 + 3 \\ 212 &= 200 + 10 + 2 \\ \underline{364} &= \underline{300 + 60 + 4} \end{aligned}$$

To show addition on the abacus, we may combine the beads column by column. Where shall we begin? **(First we combine the counters on the ones' rod: 3 and 2 are 5, and 5 and 4 are 9.)**

Next combine the counters on the tens' rod and then the counters on the hundreds' rod. Ask suitable questions so pupils understand the technique. The written record should be completed:

$$\begin{aligned} 523 &= 500 + 20 + 3 \\ 212 &= 200 + 10 + 2 \\ 364 &= \underline{300 + 60 + 4} \\ & 1000 + 90 + 9 = 1099. \end{aligned}$$

The teacher and children should discuss as many exercises as are needed to help children understand that the column form of writing an exercise helps them think about the ones together, the tens together, etc.

The children should be encouraged to line up the digits neatly as they write them in columns. Care in writing contributes to increased skill and accuracy.

The convenience of the vertical arrangement should be emphasized as means of obtaining answers efficiently. At the same time, the most important objective of this unit is to help pupils understand what they are doing.

Studying the operation of addition carefully and showing it on an abacus helps us understand the method of addition.

We need to find a method of adding quickly. We need also to be sure the sums we get are correct.

The teacher should provide exercises for the children to add in which the sum in any column is not more than 9. The pupil will add each column in order from right to left and record the sum of each column as it is added.

Addition with addends whose numerals have many digits is often needed in problem solving. Find Exercise Set 2 in your book on page 305.

(1) A salesman traveled 453 miles in January and 523 miles in February. What distance did he travel in the two months?

In this example we will follow the method which we have used before to find the answer to the questions asked in a problem.

Read example (1) carefully. What question is asked? What bits of information are given? Write the mathematical sentence which describes the problem. ($453 + 523 = n$) What operation should you use? (addition) Answer the question asked in the problem. (The salesman drove 976 miles in the two months.)

> Children should be encouraged to arrange their work like this:
>
> $$453 + 523 = n$$
>
> $$\begin{array}{r} 453 \\ +\ \underline{523} \\ 976 \end{array}$$
>
> The salesman traveled 976 miles in January and February.
>
> Exercise Set 2 may be completed now by each pupil individually.

REVIEW OF ADDITION

Exercise Set 2

1. A salesman traveled 453 miles in January and 523 miles in February. What distance did he travel in the two months? (453 + 523 = n. The salesman traveled 976 miles in the two months.)

2. The salesman traveled 230 miles in March, 310 miles in April, and 345 miles in May. How many miles did he travel in the three months? (230 + 310 + 345 = n. The salesman traveled 885 miles in the 3 months.)

3. From January through June the salesman traveled 2,010 miles. From July through December he traveled 1,854 miles. How far did he travel during the year? (2010 + 1854 = n. The salesman traveled 3,864 miles during the year.)

4. You found how far the salesman traveled in one year in exercise 3. During another year he traveled 4,013 miles. What was his mileage during the two years? (3,864 + 4,013 = n. The salesman traveled 7,877 miles in the 2 years.)

5. On an automobile trip, Fred and Carol played a game by counting station wagons and trucks they saw on the highway. Fred counted 234 station wagons and Carol counted 205 trucks. How many station wagons and trucks did they count in all? (234 + 205 = n. Fred and Carol counted 439 station wagons and trucks.)

6. Jack and Tim have been gathering rocks for the new walk their father is making. Jack has gathered 172 rocks and Tim has gathered 213. How many rocks have the two boys gathered altogether? (172 + 213 = n. The two boys gathered 385 rocks altogether.)

MORE ADDITION

Objective: To help children understand the technique for adding numbers whose numerals have many digits and their dependence on the properties of our numeration system, and the commutative property, and the associative property for addition

Materials Needed: Abacus (Place value charts may also be used.)

Exploration:

Let us find n if n = 517 + 238 + 124.

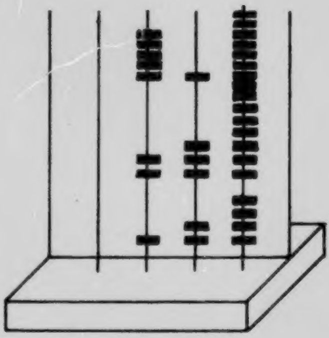
Represent the addends of this exercise on the abacus.

Write on the chalkboard:

517 = 500 + 10 + 7

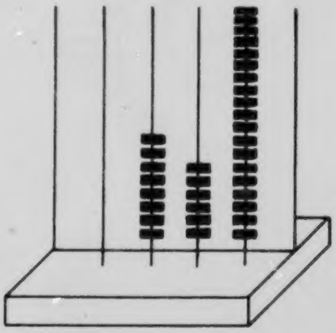
238 = 200 + 30 + 8

124 = $\underline{100 + 20 + 4}$



Tell how each numeral we wrote on the chalkboard is represented on the abacus.

Now show the addition of ones, tens, and hundreds on the abacus and on the chalkboard.



500 + 10 + 7

200 + 30 + 8

$\underline{100 + 20 + 4}$

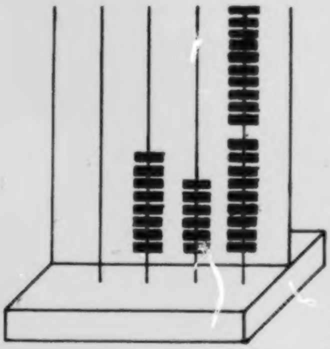
800 + 60 + 19

Is each sum 800 + 60 + 19? How can you write this as a decimal numeral?

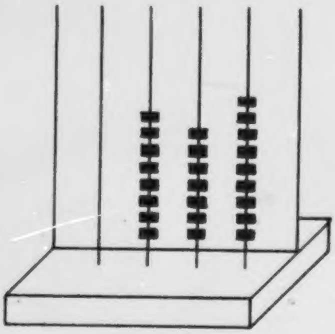
Here is a written record or our thinking.

800 + 60 + 19 =
800 + 60 + (10 + 9) = See **A** below
800 + (60 + 10) + 9 =
800 + 70 + 9 = 879. See **B** below

On the abacus show how to find the sum as a decimal numeral.



A



B

(Rearrange the 19 beads in the ones' place into 10 and 9 as in **A**. Then replace the 10 beads in the ones' place by 1 bead in the tens' place, as shown in **B**.)

It is now easy to name the sum as a decimal numeral, 879.

The teacher should use other similar exercises as needed to develop the meaning basic to the technique of addition. Analysis by showing the process in writing, on an abacus, and in discussion is helpful to the children.

Sometimes naming a sum as a decimal numeral requires more steps. One such example is 375 + 278. The teacher may wish to use the abacus to show this addition. Whether or not the abacus is used, the procedure should be examined carefully:

$$\begin{array}{l} 500 + 140 + 13 \\ 500 + 140 + (10 + 3) \\ 500 + (140 + 10) + 3 \\ 500 + (100 + 50) + 3 \\ (500 + 100) + 50 + 3 = 653\,. \end{array}$$

We do not always add using this long method. Some of us can "think the answer" without any writing. Let us study this example together.

$$\begin{array}{r} 375 \\ +\ \underline{278} \\ 13 \\ 140 \\ \underline{500} \\ 653 \end{array}$$

What is the sum of the ones? (13) Notice that we write it under the 78 in 278. In what place is the 3 in 13? (Ones' place) In what place is the 1? (Tens' place) Does 13 mean 1 ten and 3 ones? (Yes.)

What is the sum of the tens? (14) Where shall I write it? (We write 0 in the ones' place, 4 in the tens' place, and 1 in the hundreds' place.)

How can we find the sum? (Add the numbers 13, 140, and 500.) What is the sum? (653)

The above exploration is summarized on pages 306 and 307 of the Pupils' Book. It should be studied by pupils and teacher. Then, pupils may do Exercise Set 3 individually.

P306

MORE ADDITION

1. What number is n if 423 + 345 + 214 = n?

First, place beads on the abacus to show the addends so that each addend is separated from the others.

$$\begin{array}{r} 400 + 20 + 3 \\ 300 + 40 + 5 \\ \underline{200 + 10 + 4} \end{array}$$

Next, show the result of adding the ones.

Show the result of adding the tens.

Show the result of adding the hundreds.

$$\begin{array}{r} 400 + 20 + 3 \\ 300 + 40 + 5 \\ \underline{200 + 10 + 4} \\ 900 + 70 + 12 \end{array}$$

Now 423 + 345 + 214 = 900 + 70 + 12.

900 + 70 + 12 is thought of as 900 + 70 + (10 + 2).

900 + 70 + (10 + 2) = 900 + (70 + 10) + 2.

What is the decimal numeral for 900 + 80 + 2? *(982)*

2. Now try to add 342, 124 and 418 without the abacus. See **Box A.**

   a. What numbers were added first? *(the ones)*

   b. What decimal numeral is 800 + 70 + 14? *(884)*

$$\boxed{\begin{array}{rcl} 342 & = & 300 + 40 + 2 \\ 124 & = & 100 + 20 + 4 \\ \underline{418} & = & \underline{400 + 10 + 8} \\ & & 800 + 70 + 14 = 884 \end{array}}$$

A

3. Find n if 375 + 278 = n. You might try the method on page 306. **In Box B, the decimal numeral 653 was obtained from adding** 500, 140, **and** 13.

$$\boxed{\begin{array}{rcl} 375 & = & 300 + 70 + 5 \\ 278 & = & \underline{200 + 70 + 8} \\ & & 500 + 140 + 13 = 653 \end{array}}$$

B

4. Sometimes 375 and 278 are added as in box **C**. What numbers were added to get 13? (5 and 8) How do you get the 140? (add 70 and 70) How do you get 500? (add 300 and 200) How is the 653 obtained? (add 500, 140, and 13) The method of **Box C** may be more convenient for you.

$$\boxed{\begin{array}{r} 375 \\ \underline{278} \\ 13 \\ 140 \\ \underline{500} \\ 653 \end{array}}$$

C

Exercise Set 3

1. Use the method of **Box B** to find each sum.

43	167	346	558	1287
29	254	186	645	3648
(72)	(421)	(532)	(1203)	(4935)

2. Use the method of **Box C** to find each sum.

429	697	1278	8296	6278
385	134	4193	1376	1032
(814)	(831)	(5471)	(9672)	(7310)

ANOTHER METHOD FOR ADDING

Most people use a shorter method for adding. Many of you are using it already. Let us add these numbers:

$$\begin{array}{r}366\\298\\452\\\underline{647}\end{array}$$

What is the sum of the ones? (23) What is the sum of the tens? (24) What is the sum of the hundreds? (15)

We have been writing these **numbers like this:**

$$\begin{array}{r}23\\240\\\underline{1500}\\1763\end{array}$$

Then we add them to find the sum of the addends.

You can use this same method and write only part of the sum at a time and remember part of it.

> This procedure is summarized on page 308. It should be studied by the teacher and class. The children should have exercises as needed to develop the skill required in this type of addition exercise. Stress the importance of knowing the meaning basic to the technique of addition. Exercise Sets 4, 5 and 6 may be assigned for independent work at this time. Solutions to problems in Exercise Set 5 should be recorded in the form described on page 505 of this Teachers' Commentary.

ANOTHER METHOD FOR ADDING

Addition is an operation on two numbers. When we operate on 15 and 3 and get 18, we have added. (15 + 3 = 18) Eighteen is called the sum. Fifteen and 3 are called addends.

An addition exercise is written in columns to make it easy to add. Columns help to keep the ones together, the tens together, the hundreds together, and so on.

In column addition the ones are added first, the tens next, the hundreds next, and so on.

Part of the sum of the ones' column is sometimes remembered. It is then added in with the tens.

Part of the sum of the tens' column is sometimes remembered. It is then added in with the hundreds.

$$\begin{array}{r}329\\146\\\underline{948}\end{array}$$

To add I think: 9 and 6 are 15 and 15 and 8 are 23. Think of 23 as 2 tens and 3 ones. Record 3 and remember 2 tens.

Two tens and 2 tens are 4 tens; 4 tens and 4 tens are 8 tens; and 8 tens and 4 tens are 12 tens. Think of 12 tens as 1 hundred and 2 tens. Record 2 tens and remember 1 hundred.

One hundred and 3 hundreds are 4 hundreds; 4 hundreds and 1 hundred are 5 hundreds; and 5 hundreds and 9 hundreds are 14 hundreds. Record 14 hundreds.

Exercise Set 4

Find the sums for exercises 1 through 5.

	(a)	(b)	(c)	(d)	(e)	(f)
1.	43 29 (72)	57 38 (95)	19 46 (65)	76 15 (91)	68 28 (96)	53 17 (70)
2.	126 246 (372)	348 629 (977)	167 726 (893)	239 43 (282)	468 504 (972)	282 509 (791)
3.	563 128 (691)	635 406 (1041)	447 129 (576)	563 129 (692)	38 257 (295)	647 39 (686)
4.	174 138 (312)	88 543 (631)	489 272 (761)	179 658 (837)	266 698 (964)	593 248 (841)
5.	347 897 304 (1,548)	256 1297 540 698 (2,791)	1591 8643 9275 5873 (25,382)	1876 7235 8544 6718 (24,373)	8976 1235 7142 6473 (23,826)	1762 4391 3065 8572 (17,790)

6. **Find n for each of exercises (a) through (d).**

(a) n = 697 + 384 (n = 1081)

(b) n = 672 + 1278 (n = 1950)

(c) n - 559 = 2476 (n = 3035)

(d) 362 = n - 875 (n = 1237)

Exercise Set 5

1. List the number of days in each of the first six months of this year. How many days are there in the first six months of this year? (31+28+31+30+31+30=n. There are 181 days in the first six months of this year.)

2. List the number of days in each of the last six months of this year. How many days are there in the last six months of this year? (31+31+30+31+30+31=n. There are 184 days in the last six months of this year.)

3. John went to a book store. He found 5 magazines which he wanted. Their prices were 75¢, 20¢, 25¢, 55¢, and 95¢. He bought the three which were cheapest. How much did they cost? (20+25+55=n. John paid 100¢ or $1.00 for the three cheapest books.)

4. There were 135 books borrowed from the library on Monday, 140 books on Tuesday, 168 books on Wednesday, 174 books on Thursday, and 147 books on Friday. During these five days, how many books were borrowed? (135+140+168+174+147=n. There were 764 books borrowed during these five days.)

5. The Jackson family took a trip by car from New York City to Boston. The trip took five hours. This is how far they traveled each hour: 36 miles, 44 miles, 47 miles, 41 miles, and 38 miles. How many miles did they travel in the five hours? (36 + 44 + 47 + 41 + 38 = n. The Jackson family traveled 206 miles in five hours.)

6. John's mother bought him a new coat, cap, shoes, and boots. The cost of the coat was $18, the cap $3, the shoes $8, and the boots $6. How much did she pay for them all?
(18 + 3 + 8 + 6 = n. John's mother paid $35 for them all.)

7. There are 65,761 Indians in Arizona, 53,769 Indians in Oklahoma, and 41,901 Indians in New Mexico. How many Indians live in these three states? (65,761 + 53,769 + 41,901 = n. There are 161,431 Indians living in these three states.)

8. There are 629 boys and 587 girls in Longfellow School. How many children attend Longfellow School? (629 + 587 = n. There are 1216 children attending Longfellow School.)

9. In 1940 there were 172,172 people in Miami, Florida. In 1950 there were 87,063 more people living there than in 1940. How many people lived in Miami in 1950?
(172,172 + 87,063 = n. In 1950, 259,235 people lived in Miami.)

10. During a candy sale Mary sold 232 boxes of mints. Sue sold 472 boxes, and Jane sold 143 boxes. Find the total number of boxes sold by the three girls.
(232 + 472 + 143 = n. The three girls sold 847 boxes of candy.)

11. The pupils of Oak School collected gifts for poor children at Christmas. They collected 433 books, 316 toys, 252 games, and 164 puzzles. How many gifts were collected in all? (433 + 316 + 252 + 164 = n. There were 1165 gifts collected.)

P312

Exercise Set 6

Copy the numerals 1 through 8 on your paper. Next to each numeral write the words and numerals to complete this chart.

	Numbers Operated On	Result	Operation Used	Mathematical Sentence
1.	15, 289	(304)	addition	(15+289=n)
2.	139, 76	215	addition	(139+76=n)
3.	(674, 879)	n	addition	674 + 879 = n
4.	71, 56	127	(addition)	(71+56=n)
5.	641, (379+81)	(1101)	addition	(641+(379+81)=n)
6.	162, 69	(231)	addition	(162 + 69 = n)
7.	345, 187	532	(addition)	(345+187=n)
8.	647, 387	(1034)	addition	(647+387=n)

Write = or ≠ so each of exercises 9 through 15 will be true mathematical sentences.

9. 372 + 499 (≠) 773

10. 312 + 184 (≠) 128

11. 346 + n (≠) 179, if n = 177

12. n + 156 (≠) 394, if n = 328

13. n - 341 (=) 159, if n = 500

14. If n = 379, then n + 172 (≠) 308 + 233

15. If n = 473, then 896 + n (≠) 674 + 595

REVIEW OF SUBTRACTION

Objective: To help pupils review the technique of subtraction in cases where obtaining the unknown addend is a simple procedure

Exploration:

Subtraction is an operation on numbers. When we studied its properties, we operated on numbers like 8 and 5. We shall now learn a way to subtract numbers whose numerals have many digits.

Here are four **mathematical sentences.**

(a) $6 - 2 = n$ (c) $600 - 200 = n$

(b) $60 - 20 = n$ (d) $6000 - 2000 = n$

What is the sum in each? (In (a) it is 6. In (b) it is 60. In (c) it is 600. In (d) it is 6000.) What is the known addend in each? What is the unknown addend in each? (In (a) it is 4. In (b) it is 40. In (c) it is 400. In (d) it is 4000.)

Subtraction exercises like (b), (c), and (d) are often written in column form just like addition exercises. We know this is a good way to keep the ones, tens, hundreds, etc. **in the same column.**

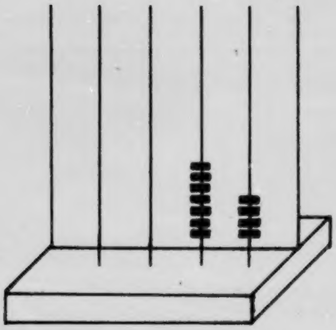
(b) $\begin{array}{r} 60 \\ -\ \underline{20} \end{array}$ (c) $\begin{array}{r} 600 \\ -\ \underline{200} \end{array}$ (d) $\begin{array}{r} 6000 \\ -\ \underline{2000} \end{array}$

How do you subtract **32 from 74?**

$\begin{array}{r} 74 \\ -\ \underline{32} \\ 42 \end{array}$

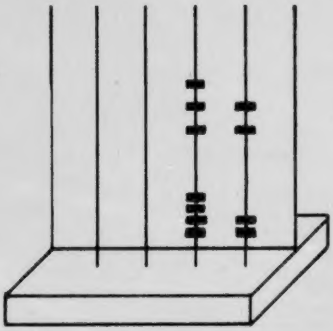
(2 subtracted from 4 is 2)
(30 subtracted from 70 is 40)
(32 is one addend; the other is 42)

How can we show this on an abacus? (We will start with 74, because it is the sum. (See I).)



I

The known addend is 32. We separate "3 tens" and "2 ones from the "7 tens" and "4 ones". How many tens and ones are there in the unknown addend? (See II) (4 tens and 2 ones) What is the **unknown addend? (42)**



II

We can show this subtraction on the chalkboard. We write the subtraction exercise in column form like this:

$$\begin{array}{rl} 74 = & 70 + 4 \\ -\underline{32} = & \underline{30 + 2} \\ & 40 + 2 = 42 \end{array}$$

Column subtraction, like column addition, helps us to think of the ones together, the tens together, etc. Here are some exercises:

$$\text{(a)}\ \begin{array}{r} 734 \\ -\ \underline{213} \end{array} \qquad \text{(b)}\ \begin{array}{r} 9400 \\ -\ \underline{3300} \end{array} \qquad \text{(c)}\ \begin{array}{r} 2640 \\ -\ \underline{1420} \end{array}$$

Tell how you would subtract in each of these exercises by thinking about ones, tens, hundreds, etc.

> Exercise Set 7, may be assigned now. Pupils should solve as many exercises of this type as are needed **in which no regrouping of the sum is necessary.**

REVIEW OF SUBTRACTION

Exercise Set 7

1. George had 524 for the answer to an exercise. It should have been 639. How much too small was his answer? (639-524 = n, or 524+n = 639. His answer was 115 too small.)

2. The zoo keeper told Jim that the big gorilla weighed 572 pounds, and the small one weighed 361 pounds. How much more does the large gorilla weigh? (572-361=n, or 361+n=572. The large gorilla weighs 211 pounds more.)

3. In 1950, the population of a city was 6,478. By 1960, it had increased to 9,699. What was the increase in population during the ten-year period? (9699-6478=n, or 6478+n = 9699. The population increased 3,221 during the ten year period.)

4. The Boy Scouts had a paper drive. Troop 51 collected 8,200 pounds of paper. They wanted to collect 9,600 pounds. How many more pounds of paper do they need to collect? (9600-8200=n, or 8200+n = 9600. Troop 51 needs to collect 1400 more pounds of paper.)

5. Subtract

$$\begin{array}{cccc} 665 & 841 & 937 & 269 \\ \underline{152} & \underline{721} & \underline{125} & \underline{253} \\ (513) & (120) & (812) & (16) \end{array}$$

6. Find n so each mathematical sentence will be true.

(a) $n + 395 = 697$ $(n=302)$ (c) $863 + n = 1175$ $(n=312)$

(b) $n = 1158 - 737$ $(n=421)$ (d) $2378 - 2163 = n$ $(n=215)$

MORE SUBTRACTION

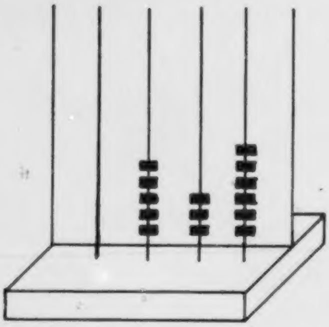
Objective: To help children understand the technique for subtracting numbers whose numerals have many digits and its dependence on the properties of our numeration system

Materials Needed: Abacus

Exploration:

I have this problem which we should study together: "Janet read 536 pages from library books in one month and Emily read 218 pages. How many more pages must Emily read to have read as many pages as Janet?"

Study the problem carefully. What is the question to be answered? What are the bits of information in the problem? Write a mathematical sentence for the problem situation. (218 + p = 536 or p = 536 - 218) **What operation is needed to answer the question?** (Subtraction) **Write the subtraction in column form. Show this subtraction on an abacus.** (See I)



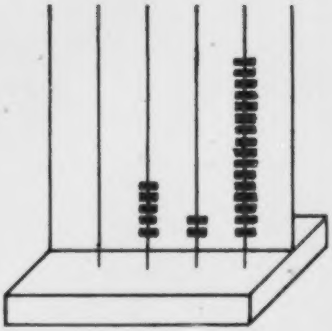
I

$$\begin{array}{r} 536 \\ \underline{218} \end{array}$$

Show how to express the sum and known addend on the chalkboard.

$$\begin{array}{r} 536 = 500 + 30 + 6 \\ \underline{218} = 200 + 10 + 8 \end{array}$$

Can we separate the known addend from the sum on the abacus? (No, because the sum has too few markers in the ones' place.) Show on the abacus how we can regroup the sum to get more ones. (Change it to 5 hundreds, 2 tens, and 16 ones.)



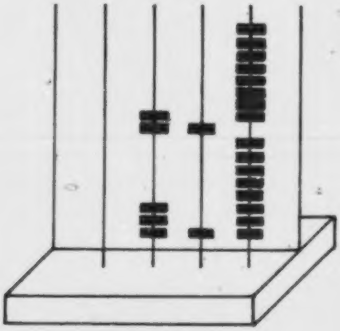
II

Show this on the chalkboard.

$$\begin{aligned} 536 &= 500 + 30 + 6 = 500 + 20 + 16 \\ \underline{218} &= \underline{200 + 10 + 8} = \underline{200 + 10 + 8} \end{aligned}$$

Can the known addend now be separated from the sum on the abacus? (Yes) Show this. (See III)

What is the other addend? (318)



III

Can we show on the chalkboard how the subtraction is done? (Yes) Describe the subtraction. (16 - 8 = 8. 20 - 10 = 10. 500 - 200 = 300. The 8 is written in the ones place, 1 is written in the tens place, and 3 is written in the hundreds place. The other addend is 318.)

$$536 = 500 + 30 + 6 = 500 + 20 + 16$$

$$\underline{218} = \underline{200 + 10 + 8} = \underline{200 + 10 + \ \ 8}$$

$$300 + 10 + \ \ 8 = 318$$

The teacher should provide as many exercises as are needed to develop the meaning basic to the technique of subtraction. The **steps should be** shown in writing and on an abacus as needed.

A summary of the above exploration is found in **exercises** 1 and 2 of **Pupils' Book,** pages 314 and 315. It should be studied by the teacher and class together.

Sometimes naming the unknown addend as a decimal numeral is more difficult. An example and a method for completing the subtraction are described in **exercise** 3, page 315. It should be carefully studied and discussed by the teacher and class. No further description is included here. Speed in completing such a procedure is not important and extensive drill is not recommended. At the same time, the use of this method for selected exercises now and at later times during the year is recommended.

MORE SUBTRACTION

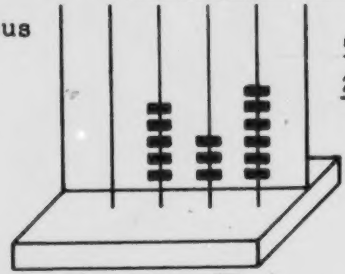
Subtraction is an operation for finding the unknown addend if the sum and one addend are known. To find 536 - 218, you have learned to write as shown in box A.

$$\begin{array}{lll} \text{A} & 536 & \text{sum} \\ & \underline{218} & \text{addend} \\ & ? & \text{unknown addend} \end{array}$$

1. Could you subtract using the form of box B? Why?

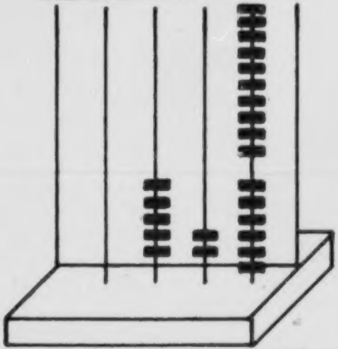
$$\begin{array}{ll} \text{B} & 536 = 500 + 30 + 6 \\ & \underline{218} = \underline{200 + 10 + 8} \end{array}$$

Now, let us use the abacus to help us think about this process. First we show the sum 536 on the abacus



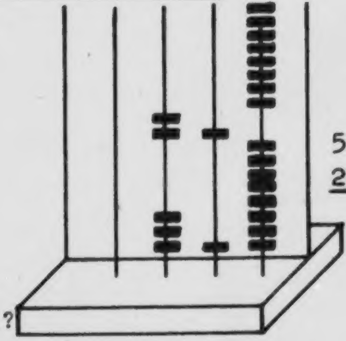
$$\begin{array}{l} 500 + 30 + 6 \\ \underline{200 + 10 + 8} \end{array}$$

Then, we think of the sum 536 as 500 + 20 + 16



$$\begin{array}{l} 500 + 20 + 16 \\ \underline{200 + 10 + \ \ 8} \end{array}$$

Now, we separate the markers to show the known addend, 218, and the other addend. What is the other addend?



$$\begin{array}{l} 500 + 20 + 16 \\ \underline{200 + 10 + \ \ 8} \end{array}$$

2. The written record of the above subtraction is

$$\begin{array}{lclcl}
536 & = & 500 + 30 + 6 & = & 500 + 20 + 16 \\
\underline{218} & = & \underline{200 + 10 + 8} & = & \underline{200 + 10 + \ \ 8} \\
 & & & & 300 + 10 + \ \ 8 \ = \ 318
\end{array}$$

3. Sometimes, finding the unknown addend is more difficult.

For example, what is n, if $268 + n = 932$?

We may write:

$$\begin{array}{lclclcl}
932 & = & 900 + 30 + 2 & = & 900 + 20 + 12 & = & 800 + 120 + 12 \\
\underline{268} & = & \underline{200 + 60 + 8} & = & \underline{200 + 60 + \ \ 8} & = & \underline{200 + \ \ 60 + \ \ 8} \\
 & & & & & & 600 + \ \ 60 + \ \ 4 = 664
\end{array}$$

Explain how we may think when subtracting in this way.

What is the other addend?

Now let us look for a shorter way of writing the steps in a subtraction problem. Notice how this form corresponds to the one above. We begin with

$$\begin{array}{rl}
932 & \text{9 hundreds, 3 tens, 2 ones} \\
\underline{-\ 268} & \text{2 hundreds, 6 tens, 8 ones.}
\end{array}$$

We cannot subtract in the ones' column so we regroup

$$\begin{array}{ccccl}
 & \textcircled{2} & \textcircled{12} & & \\
9 & \cancel{3} & 2 & & \text{9 hundreds, 2 tens, 12 ones} \\
2 & 6 & 8 & & \text{2 hundreds, 6 tens, 8 ones.}
\end{array}$$

We cannot subtract in the tens' column so we regroup again

$$\begin{array}{ccccl}
\textcircled{8} & \textcircled{12} & \textcircled{12} & & \\
\cancel{9} & \cancel{3} & \cancel{2} & & \text{8 hundreds, 12 tens, 12 ones} \\
\underline{-\ 2} & \underline{6} & \underline{8} & & \text{2 hundreds, 6 tens, 8 ones} \\
6 & 6 & 4 & &
\end{array}$$

$12 - 8 = 4$, 4 **ones**

$12 - 6 = 6$, 6 **tens**

$8 - 2 = 6$, 6 **hundreds**

ANOTHER METHOD FOR SUBTRACTING

There is a shorter method of subtraction which most people use. You can use it too. It is very much like the short method of addition. It requires you to think of the convenient name for the sum instead of writing it. Think of these exercises.

(a) $\begin{array}{r} 75 \\ -\ \underline{23} \end{array}$ (b) $\begin{array}{r} 41 \\ -\ \underline{23} \end{array}$ (c) $\begin{array}{r} 58 \\ -\ \underline{23} \end{array}$

Can 23 in exercise (a) be subtracted directly as the sum is named? (Yes) In (b)? (No) In (c)? (Yes)

Let us study exercise (b). Without doing any writing, can you think of 4 tens and 1 **one so that 2 tens and 3 ones may be subtracted? (Yes, I think 3 tens + 11.)**

Remember this and subtract. Describe what you think. (11 - 3 = 8. **3 tens - 2 tens is 1 ten, or 10. The result is 18.)**

Try these just by thinking about the convenient name for the sum.

(d) $\begin{array}{r} 43 \\ -\ \underline{28} \end{array}$ (e) $\begin{array}{r} 75 \\ -\ \underline{36} \end{array}$ (f) $\begin{array}{r} 424 \\ -\ \underline{162} \end{array}$ (g) $\begin{array}{r} 424 \\ -\ \underline{248} \end{array}$

Answer this question for each: How can we think of the sum to subtract the known addend?

(d) 43 = 30 + 13.
(e) 75 = 60 + 15.
(f) 424 = 300 + 120 + 4.
(g) 424 = 300 + 110 + 14.

> Pupils are not expected to be able to supply an immediate correct answer to exercises such as (g) above. To help pupils understand what they are doing is the objective of this development.

If they can write and tell reasons for the thinking below, the teacher should believe that they understand the technique.

$$424 = 400 + 20 + 4 = 400 + 10 + 14 = 300 + 110 + 14$$

$$\underline{248} = \underline{200 + 40 + 8} = \underline{200 + 40 + 8} = 200 + 40 + 8$$

The teacher and class should read and study page 316. Exercise **Sets 8 and 9 may be assigned** for pupils to complete independently. The pupils' work for solving the problems in Exercise Set 9 should be arranged as described on page 505 of this commentary.

Special attention is given in Exercise Set 10 to subtractions such as 800 - 342 in which one or more 0's are in the name of the sum.

Using only tens, hundreds, thousands, etc. tell me different names for:

400 (4 hundred, or 40 tens)

1000 (1 thousand, or 10 hundreds or 100 tens)

Here are some subtractions. How should we think of the sum so we can find the unknown addend easily?

(a)	(b)	(c)	(d)
6000	6000	6000	6000
- 2000	- 2300	- 2340	- 2345

(a) 6000 = 6 **thousands**

(b) 6000 = 5 **thousands + 10 hundreds**

(c) 6000 = 59 **hundreds + 10 tens**

(d) 6000 = 599 tens + 10 ones

Make a written record of subtractions (b), (c) and (d):

(b)

$$\begin{array}{rcl} 6000 & = & 5000 + 1000 \\ 2300 & = & \underline{2000 + 300} \\ & & 3000 + 700 = 3{,}700 \end{array}$$

(c)

$$\begin{array}{rcl} 6000 & = & 5900 + 100 \\ 2340 & = & \underline{2300 + 40} \\ & & 3600 + 60 = 3{,}660 \end{array}$$

(d)

$$\begin{array}{rcl} 6000 & = & 5990 + 10 \\ 2345 & = & \underline{2340 + 5} \\ & & 3650 + 5 = 3655 \end{array}$$

In order to obtain correct answers to (a), (b), (c), and (d) above, pupils may need to experiment with different names for the sum. There are other ways of renaming 6000 in (d); for example 5 thousands + 9 hundreds + 9 tens + 10 ones. The name 599 tens + 10 ones is simpler form.

This exploration is summarized on pupils' page 319. It should be studied and discussed by the teacher and class. Exercise Sets 10 and 11 may be assigned as independent work.

An algorism for subtraction is given on page 319. In the regrouping the pupils will soon learn to do all regrouping at one step. For example,

$$\begin{array}{cccc} & & \textcircled{12} & \\ & \textcircled{8} & \cancel{\textcircled{2}} & \textcircled{12} \\ & \cancel{9} & \cancel{3} & \cancel{2} \\ - & 2 & 6 & 8 \\ \hline & 6 & 6 & 4 \end{array}$$

ANOTHER METHOD FOR SUBTRACTING

Subtraction is an operation on numbers. When we operate on 15 and 3 and get 12, we have subtracted. 15 - 3 = 12. And, 12 is called the unknown addend.

A subtraction exercise is written in columns to make subtraction easy. Columns help to keep the ones together, the tens together, etc.

In column subtraction the ones are subtracted first, the tens next, etc.

Renaming the sum in a subtraction exercise may help us to subtract.

  5576
- 1328

To subtract I think: There are not enough ones in the ones' place in 5,576. I will think of 5,576 as 5 thousands, 5 hundreds, 6 tens, and 16 ones. 16 - 8 = 8. 6 - 2 = 4. 5 - 3 = 2. 5 - 1 = 4. The unknown addend is 4,248.

Exercise Set 8

Find the unknown addend in each of 1 and 2.

	(a)	(b)	(c)	(d)	(e)
1.	93 38 (55)	187 99 (88)	817 748 (69)	852 575 (277)	596 378 (218)
2.	5634 1256 (4378)	2876 259 (2617)	8421 5167 (3254)	3124 2674 (450)	5672 1489 (4183)

Exercise Set 9

1. One week a factory assembled 2,640 trucks and 1,582 automobiles. How many more trucks than automobiles were assembled? (2,640 - 1,582 = n, or 1,582 + n = 2,640. There were 1058 more trucks than automobiles assembled.)

2. In 1950 there were 3,500 people in Woodside. In 1960 there were 9,400 people in Woodside. How many more people were there in 1960 than in 1950? (9400 - 3500 = n, or 3500 + n = 9400. There were 5900 more people in 1960.)

3. We planned a 455 mile trip. The first day we traveled 266 miles. How many miles were left to travel? (455 - 266 = n, or 266 + n = 455. There were 189 miles left to travel.)

4. The Mississippi River is 2,348 miles long and the Ohio River is 981 miles long. How many miles longer is the Mississippi River? (2348 - 981 = n, or 981 + n = 2348. The Mississippi River is 1367 miles longer.)

5. What is the total length of the Mississippi and the Ohio Rivers? (2348 + 981 = n. The total length is 3329 miles.)

6. In New York City, the Empire State Building is 1,472 feet high. The Chrysler Building is 1,046 feet high. How much higher is the Empire State Building? (1472 - 1046 = n, or 1046 + n = 1472. The Empire State Building is 426 feet higher than the Chrysler Building.)

7. There were 435 children at Whittier School and 379 children at Edison School. How many children attend both schools? ($435 + 379 = n$. There are 814 children in both schools.)

8. A sign on a foot bridge reads, "Not safe for over 200 pounds." Jerry weighs 62 pounds, Dick weighs 57 pounds, Tom weighs 68 pounds. Can the three boys safely walk across the bridge together? ($62 + 57 + 68 = n$. The three boys weigh 187 pounds together. Since 187 is less than 200, the three boys can walk across the bridge safely.)

9. Another bridge holds two tons safely. A cement truck that weighs 2,165 pounds is on the bridge. How many more pounds could safely be on the bridge at the same time? ($2{,}165 + n = 4{,}000$, or $4{,}000 - 2{,}165 = n$. 1835 more pounds could safely be on the bridge at the same time.)

10. Susan's grandmother was born in 1908. How old will she be on her birthday this year? ($1962 - 1908 = n$, or $1908 + n = 1962$. Susan's grandmother will be 54 on her birthday in 1962.)

SUBTRACTION WITH ZEROS

1. Which of these are other names for 8,000? (all except e)

(a) 8,000 ones   (d) 7,000 + 1,000   (g) 8 thousands
(b) 8021 - 21   (e) **800 hundreds**   (h) 8,000 - 0
(c) 800 tens   (f) 10,000 - 2,000

2. Suppose you are to find n when 8,000 - 1,732 = n. You can write the example as in **Box A**. Finding the unknown addend is easy if you rename 8,000 as 799 tens and 10 ones or 7990 + 10.

A

$$\begin{array}{r} 8000 \\ \underline{-\ 1732} \end{array}$$

3. (a) Look at the example given in **Box B**.
   (b) Tell how to get the unknown addend, 6260 + 8.
   (c) What decimal numeral names the unknown addend?

B

$$\begin{array}{rcrcr} 8000 & = & 7990 & + & 10 \\ \underline{1732} & \underline{=} & \underline{1730} & \underline{+} & \underline{2} \\ & & 6260 & + & 8 \end{array}$$

Exercise Set 10

Find the unknown addend for each of these.

	(a)	(b)	(c)	(d)
1.	804 267 (537)	602 536 (66)	102 85 (17)	3001 1467 (1534)
2.	6000 1234 (4766)	3007 1562 (1445)	4803 1297 (3506)	2067 1982 (85)

Exercise Set 11

1. John is 52 inches tall; his father is 70 inches tall. How many inches must John grow to be as tall as his father? ($52 + n = 70$, or $70 - 52 = n$. John must grow 18 inches to be as tall as his father.)

2. At Glenn School there are 500 boys and 375 girls. How many more boys are there than girls? ($500 - 375 = n$, or $375 + n = 500$. There are 125 more boys than girls.)

3. Don has 1,500 stamps. He pasted 323 in his album. How many are left to put in the album? ($1500 - 323 = n$, or $323 + n = 1500$. Don has 1177 stamps left to put in the album.)

4. Sue has $25. She is saving to buy a bicycle which costs $42. How much more money must she save? ($25 + n = 42$, or $42 - 25 = n$. Sue must save $17 more to buy a bicycle.)

5. A high school stadium has 5,200 seats. 3,482 tickets have been sold for a game. How many tickets are left? ($5200 - 3482 = n$, or $3482 + n = 5200$. There are 1718 tickets left.)

6. An elephant in a zoo weighs 5,000 pounds. A bear weighs 746 pounds. How much less does the bear weigh than the elephant? ($5,000 - 746 = n$, or $746 + n = 5,000$. The bear weighs 4,254 pounds less than the elephant.)

7. West Virginia became a state in 1863. Hawaii became a state in 1960. How many more years has West Virginia been a state than Hawaii? ($1863 + n = 1960$, or $1960 - 1863 = n$. West Virginia has been a state 97 years longer than Hawaii.)

RELATION OF THE TECHNIQUES OF ADDITION AND SUBTRACTION

Objective: To help children extend their understanding that addition of a number and subtraction of that same number undo each other.

Material Needed: Abacus

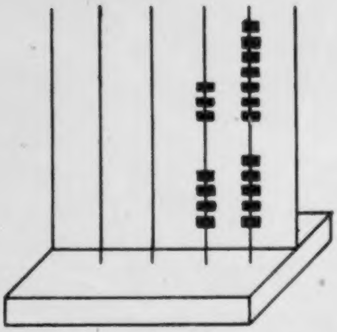
Exploration:

We learned that addition of a number and subtraction of that same number undo each other. Let us see if we can show this on an abacus with this example.

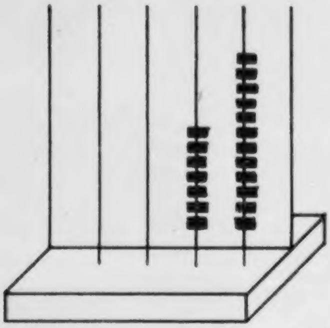
Add:		Subtract:	
37	(Addend)	82	(Sum)
45	(Addend)	45	(Addend)
82	(Sum)	37	(Addend)

> The children will find it helpful to use an abacus to sense clearly that the thinking associated with "combining" and "separating" markers illustrates the idea of doing and undoing. On page 536 Column I pictures 37 + 45 = 82 on the abacus. Column II pictures 82 - 45 = 37 on the abacus. Similarly we could picture on the abacus 45 + 37 = 82 and 82 - 37 = 45.
>
> Now let us think about the addition and the subtraction without an abacus.
>
> | Add | Subtract |
> |---|---|
> | 37 | 82 |
> | 45 | 45 |
> | 82 | 37 |

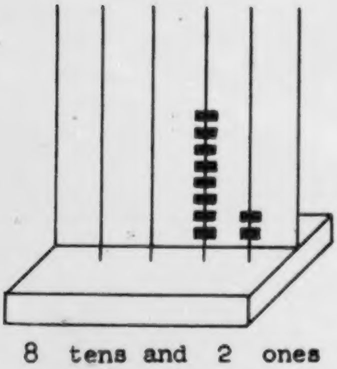
Column I



37 and 45

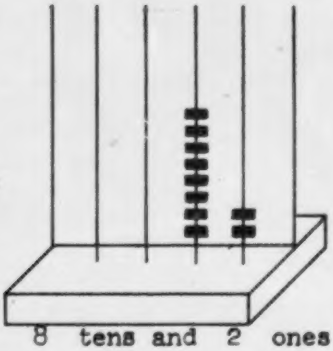


7 tens and 12 ones

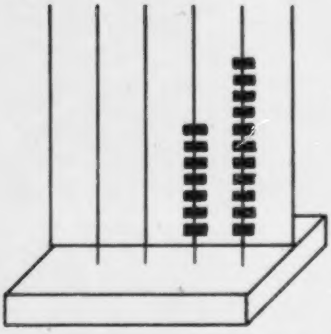


8 tens and 2 ones

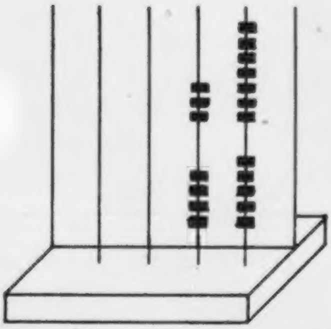
Column II



8 tens and 2 ones



7 tens and 12 ones



8 tens and 2 ones

Now have children explain how subtraction of a number undoes addition of that same number. They find the sum of two addends. Subtraction of either addend from the sum gives the other addend.

Exercise Set 12 may now be assigned.

Exercise Set 13 is a set of mixed practice. These should be assigned to pupils who need such practice.

RELATION OF THE TECHNIQUES OF ADDITION AND SUBTRACTION

Exercise Set 12

Copy the chart below. Add or subtract each exercise and then undo each.

Do	Undo
1. Add 725 342 (1067)	1. (Subtract 1067 - 342 725 or 1067 - 725 342)
2. Subtract 1629 817 (812)	2. (Add 817 812 1629)
3. Subtract 5232 768 (4464)	3. (Add 768 4464 5232)
4. Add 5287 9388 (14,675)	4 (Subtract 14,675 9,388 5,287 or 14,675 5,287 9,388)
5. Add 26534 12986 (39,520)	5. (Subtract 39,520 12,986 26,534 or 39,520 26,534 12,986)

6. Show that each of these mathematical sentences about doing and undoing is true. The first one is done for you as an example.

(a) (573 + 128) - 128 = 573.

Answer:	573	701
	128	128
	701	573

(b) (841 + 368) - 368 = 841

(c) (632 - 257) + 257 = 632

(d) (905 - 496) + 496 = 905

(e) (384 + 769) - 769 = 384

Answers

b. 841 + 368 = 1209 1209 − 368 = 841

c. 632 − 257 = 375 375 + 257 = 632

d. 905 − 496 = 409 409 + 496 = 905

e. 384 + 769 = 1153 1153 − 769 = 384

7. Column addition may be checked by using the commutative and associative properties of addition. In this example, first "add from the top down." Then "add from the bottom up." Are the sums the same? (yes) Add:

43
32
$\underline{57}$
(132)

Add and check the sums in each of the following exercises:

8.
72
49
36
42
$\underline{88}$
(287)

9.
324
964
322
$\underline{508}$
(2118)

10.
3286
9246
3078
$\underline{5000}$
(20,610)

11.
17208
15363
$\underline{42630}$
(75,201)

12.
1492
3876
9547
3841
$\underline{2056}$
(20,812)

13.
687
941
600
817
$\underline{932}$
(3977)

14.
15618
29832
75490
61078
$\underline{70201}$
(252,219)

15.
61429
78503
59268
68107
$\underline{91030}$
(358,337)

16. BRAINTWISTER: **Try to find the sum for exercise 8 by adding down the column once.**

Exercise Set 13

Copy the numerals 1 through 7 on your paper. Write the correct words or numerals to complete this chart.

	Numbers Operated On	Result	Operation	Mathematical Sentence
1.	394, 869	(1263)	addition	$(394 + 869 = n)$
2.	(762, 187)	(575)	(subtraction)	$762 - (187) = 575$
3.	498, (779)	1277	addition	$(498 + n = 1277)$
4.	(297 + 356), 495	(158)	subtraction	$((297 + 356) - 495 = n)$
5.	2000, (156 + 354)	(1490)	subtraction	$(2000 - (156 + 354) = n)$
6.	(392 + 867), 201	(1058)	subtraction	$((392 + 867) - 201 = n)$
7.	(384, 979)	(1363)	(addition)	$384 + 979 = (n)$

In exercises 8 to 16, what is n so each mathematical sentence will be true?

8. $n = 67 + 48$ $(n = 115)$
9. $n = 204 - 157$ $(n = 47)$
10. $n = 4000 - 1963$ $(n = 2037)$
11. $n + 42 = 89$ $(n = 47)$
12. $102 - n = 3$ $(n = 99)$
13. $n - 128 = 568$ $(n = 696)$
14. $n + 392 = 691$ $(n = 299)$
15. $601 - n = 399$ $(n = 202)$
16. $893 - n = 256$ $(n = 637)$

17. BRAINTWISTER. In each exercise below, the letters A, B, C, D and E are to be replaced by one of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 or 9. They may be replaced by different digits in different exercises. A symbol such as AB represents a 2-place numeral.

47 + 4 A1 (51)	63 - A (5) B8 (58)	47 + D (8) CC (55)	DD (77) + E (3) 80	CC (55) + C (5) 60
ABC (195) - 42 153	E4 (54) + 3C (33) 87	BB (66) - B2 (62) 4	7A (76) - AB (62) 14	CD (16) + D3 (63) 79

THE LANGUAGE OF SUBTRACTION PROBLEMS

Objective: To help children identify the language of problems whose mathematical solution may be obtained by subtraction

Exploration:

> Children should be aware of the different problem situations in which subtraction may be used to find the solution. The operation of subtraction is used to find the missing addend in situations such as those in which we are to: (1) compare two amounts; (2) find how much more is needed; (3) find how much is left, etc.

How can you tell when you should subtract to solve a problem? Before you answer we should study some problems on page 325 of your book.

(1) **Family A traveled 323 miles and family B traveled 289 miles on a week-end trip. How many more miles did family A travel than family B?**

(2) **Family A traveled 323 miles and family B traveled 289 miles on a week-end trip. How far did they travel together?**

(3) Family A traveled 323 miles and family B traveled 289 miles on a week-end trip. How much farther would family B have to travel **in order to travel as far as family A?**

(4) Families A and B are together on a trip of 323 miles. They have traveled 289 miles. How many miles do they have left to travel?

Look for the question asked in each problem. Then find the information that is given in the problem and write a mathematical sentence using the information.

Let us solve each of these, following the method we have been using.

(1) $n = 323 - 289$ or $289 + n = 323$

$$\begin{array}{r} 323 \\ -\ \underline{289} \\ 34 \end{array}$$

Family A traveled 34 miles farther than family B.

(2) $m = 323 + 289$

$$\begin{array}{r} 323 \\ +\ \underline{289} \\ 612 \end{array}$$

The two families traveled 612 miles.

(3) $289 + p = 323$
or
$p = 323 - 289$

$$\begin{array}{r} 323 \\ -\ \underline{289} \\ 34 \end{array}$$

Family B would have to travel 34 miles to have traveled as far as Family A.

(4) $s = 323 - 289$
or
$289 + s = 323$

$$\begin{array}{r} 323 \\ -\ \underline{289} \\ 34 \end{array}$$

Families A and B have 34 miles left to travel.

How are the problems similar? (The numbers in the problems are the same.) Are the problems the same? (No, they are very different.) How do they differ? **(Different questions are asked.)**

Let us examine the work you did to answer the questions. Was there a difference in the operation used? (Yes. In (2) we added; in the others we subtracted.) How did you know which operation to use? (We could tell from the relationship in the problem.)

We can describe an addition problem as one which gives two or more addends and asks us to find their sum. Did problem (2) do these things? (Yes, it gave the addends, 323 and 289. It asked how far the two families traveled altogether.)

How did you know to subtract in the other problems? (We could tell from the relationship in the problem.) How may we describe a subtraction problem? (We know two numbers, one is the sum and the other is an addend. We are to find the unknown addend.) In problems (1), (3) and (4), which number is the sum and which is the known addend? (In problems (1), (3), and (4) the sum is 323 and 289 is a known addend.)

We have described a subtraction problem as giving a sum and known addend. We have said that problems (1), (3), and (4) have the same sum and addend given. In this way the problems are all alike even though they ask different questions.

> The children should make up a few problems requiring subtraction. Some problems should ask that amounts be compared. Other problems should be based on situations where one set is to be separated into two subsets.
>
> Exercise Set 14 may be assigned now. Pupils should use the form shown on page 505 of this commentary to record their solutions.

THE LANGUAGE OF SUBTRACTION PROBLEMS

1. **Family A traveled 323 miles and family B traveled 289 miles on a weekend trip. How many more miles did family A travel than family B?**

2. **Family A traveled 323 miles and family B traveled 289 miles on a weekend trip. How far did the two families travel?**

3. Family A traveled 323 miles and family B traveled 289 miles on a weekend trip. How much farther would family B have to travel in order to travel as far as family A?

4. Families A and B are together on a trip of 323 miles. They have traveled 289 miles. How many miles do they have left to travel?

Exercise Set 14

1. A notebook costs 15¢, a pencil 27¢, and an eraser 5¢. **How much will it cost to buy a set of one of each?**
(15+27+5 = n. It will cost 47¢ to buy a set of one of each.)

2. Four children put their savings together to help buy a riding horse. Mary had $35, Jerry had $48, Diane had $123, and Frank had $97. **How much money did the four children have?** (35+48+123+97 = n. The four children had $303.)

3. A football playing field is 300 feet long and 160 feet wide. How far will you have walked if you walk along the four edges of the field? (300+160+300+160=n. You would walk 920 feet if you walked along the edges of the field.)

4. John has 268 postage stamps. He received some for Christmas. Then he had 323. How many stamps did he receive for Christmas? (268+n=323, or 323-268=n. John received 55 stamps for Christmas.)

5. At Fairview, the temperature was 58° at noon and 23° at midnight. How much had the temperature changed? (58-23=n, or 23+n=58. The temperature changed 35°.)

6. Tom wanted to buy a radio which was priced at $72. He had $56 saved. How much did he still have to save? (56+n=72, or 72-56=n. Tom still needs to save $16 to buy the radio.)

7. On a page in a catalog the following prices were given: soft ball, $1; bat, $3; fielder's mitt, $3; catcher's mitt, $12; first baseman's mitt, $9; catcher's mask, $4; and baseball uniform, $6. What will it cost Mr. Thompson to buy a ball, a bat, and three uniforms for his sons? (1+3+6+6+6=n. It will cost Mr. Thompson $22.)

8. In one year the Acme Motor Company made 969,732 automobiles, 95,060 trucks, and 17,747 motor scooters. Find the number of vehicles made by the Acme Motor Company in that year. (969,732+95,060+17,747=n. The Acme Motor Company made 1,082,539 vehicles in that year.)

IF-THEN THINKING

Objective: To introduce to pupils one of the procedures for drawing valid conclusions--"If-then" thinking

Vocabulary: If-then thinking

Exploration:

The exploration for this topic is in the **Pupils' Book,** page 327. The teacher should emphasize for pupils the importance of "if-then" thinking. The mathematician assumes certain relationships; he states these in the "if part" of the statement. He then draws valid conclusions which he includes in the "then part" of the statement. More examples of correct and incorrect "if-then" reasoning should be provided by the teacher and following that, pupils may offer their own. Both social and mathematical statements should be used.

If a pupil says "If $n + 9 = 15$, then $9 + n = 15$," or "If $n + 9 = 15$, then $15 - n = 9$," the teacher should ask, "Give me a reason for your statement."

Assign Exercise Set 15 for independent work. After Exercise Set 15 is completed, Set 16 may be assigned. Magic squares of the latter set provide a means of **interesting practice.**

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IF-THEN THINKING

1. We often use "if-then" reasoning. For example, you may think:

   "If I run home, then I will get there quicker." or
   "If it rains, then we cannot play baseball."

   Tell some "if-then" statements about your activities.

2. In our if-then statements we want the second part to be true because of the first part.

3. We use "if-then" thinking when we reason:

   "If $7 + n = 15$, then $n + 7 = 15$" or
   "if $7 + n = 15$, then $n = 8$"

   We would not think

   "If $3 + 6 = 9$, then $3 + 6 = 10$ since the "then" part is not a result of the "if" part.

   We could think, "If $3 + 6 = 9$, then $3 + 7 = 10$."

   Complete this statement in other ways, If $3 + 6 = 9$, then . . . ($6+3=9$, $3+7=10$, etc.)

4. (a) Is it true that "If $n + 6 = 15$, then $n = 15 - 6$"? (yes)
   (b) Is it true that "If $n - 6 = 10$, then $n = 10 + 6$"? (yes)

Exercise Set 15

1. Complete these statements. **Use some different ways to complete each as you can.** (Answers will vary.)

(a) If $15 - 9 = n$, then ... (d) If $11 + n = 25$, then ...
(b) If $13 + n = 21$, then ... (e) If $12 + n = 19$, then ...
(c) If $33 = 17 + n$, then ... (f) If $n - 15 = 14$, then ...

2. Use $=$, $>$ or $<$ so each of these statements will be true.

(a) If $n + 6 = 17$, then n (<) 17.
(b) If $21 - n = 19$, then n (<) 21.
(c) If $44 = n + 27$, then n (<) 44.
(d) If $n - 16 = 31$, then n (>) 16.
(e) If $n + n = 40$, then n (<) 40.
(f) If $n + 0 = 178$, then n (=) 178.
(g) If $0 - n = 0$, then n (=) 0.
(h) If $(6 + 8) + n = 19$, then n (<) 19.

3. BRAINTWISTER. Remember: x, y, and z represent whole numbers. Suppose $x + y = z$.

(a) Are you sure that $x < z$ and $y < z$? (yes, except when $x = 0$, $y = 0$, and $z = 0$)
(b) Give an example for $x = z$. (If $x = z$, then $y = 0$.)
(c) Give an example for $x < z$. (If $x < z$, then $y < z$. Answers will vary.)
(d) **Give one example for** $x < z$ and $y < z$. (Answers will vary.)
(e) Could $x > z$? (No)

Exercise Set 16

The arrangement of numbers in the square at the right is called a magic square.

	A	B	C
D	4	3	8
E	9	5	1
F	2	7	6

1. What is the sum of the numbers in column A? (15) in column B? (15) in column C? (15)

2. What is the sum of the numbers in row D? (15) row E? (15) row F? (15)

3. The 4, 5, and 6 are said to be on a diagonal. What is their sum? (15) What three other numerals are on a diagonal? (8,5,2) What is their sum? (15)

4. Are all eight sums the same? (yes) The square is said to be "magic" because the sums of all rows, columns, and diagonals are equal.

5. Make a new square by adding 19 to each number in the above square. What is the sum of the numbers in: each row? (72) each column? (72) each diagonal? (72) Is the new square a magic square? (yes)

23	22	27
28	24	20
21	26	27

6. Is the square on the right a magic square? (yes) What is the sum of the numbers on each row, column, and diagonal? (254)

71	57	58	68
60	66	65	63
64	62	61	67
59	69	70	56

7. Make a new square by subtracting 49 from each number in the square in exercise 6. Is it a magic square? (yes

22	8	9	19
11	17	16	14
15	13	12	18
10	20	21	7

)

REVIEW

Objective: To help pupils review the meanings, skills and procedures for problem solving developed in this unit

Teaching Procedures:

Three Exercise Sets are included in this section. Set 17 is a review of skills with **Braintwisters** to provide a challenge for fast learners. Set 18 consists of word problems and braintwisters. Methods of solving problems which have been discussed earlier should be used.

Exercise Set 19 has as its specific purpose helping pupils choose the essential information from a paragraph and use it to answer a question. Further, in certain exercises the mathematical sentence is given and the pupil is to state the question which is answered by that sentence. This is the reverse of his previous experience in which he is asked a question and told to write the mathematical sentence describing it. To further this objective, pupils may be asked to make up problems requiring addition or subtraction.

All pupils need not solve every exercise or problem. The large number of exercises and problems permits the teacher to make assignments suitable to the ability of each pupil. At the same time, these sets are not completely a review. There are many variations of the content studied previously. The teacher should have some class discussion of the difficult exercises and methods for attacking them.

Notes on Braintwisters:

1. Ex. 7, page 331 Pupils should try replacing n by various numbers in $n + n = 200$ and $n + n = 582$.
2. Ex. 10, page 331 You can find n in the mathematical sentence $n - 376 = 89$ because 376 and 89 are addends; addition is always possible within the set of whole numbers. However, for $n + 376 = 89$, there is no whole number for n, because $n = 89 - 376$. Subtraction is not always possible within the set of whole numbers.

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Exercise Set 17

1. Subtract

84	126	536	1427	1674
57	67	239	1148	555
(27)	(59)	(297)	(279)	(1119)

2. Add

67	134	257	3782	2841
84	29	489	6356	7159
(151)	(163)	(746)	(10,138)	(10,000)

3. Find n so each mathematical sentence will be true.

(a) $81 - 46 = n$ (n = 35) (c) $n + 126 = 253$ (n = 127) (e) $359 - n = 284$ (n = 75)
(b) $n = 76 + 49$ (n = 125) (d) $n - 87 = 123$ (n = 210) (f) $283 + n = 481$ (n = 198)

4. Which of these mathematical sentences are not true?

(a) $81 + 69 = 160$ (F) (d) $1276 - 493 = 783$ (T)
(b) $124 + 238 = 362$ (T) (e) $263 = 612 - 350$ (F)
(c) $289 + 463 = 752$ (T) (f) $412 = 913 - 571$ (F)

5. Write =, > or < so each mathematical sentence will be true.

(a) $825 \underline{(=)} 568 + 257$ (c) $742 - 367 \underline{(>)} 374$
(b) $289 + 482 \underline{(>)} 761$ (d) $538 - 289 \underline{(<)} 259$

6. BRAINTWISTER. What whole number, if any, can be used for n so each mathematical sentence will be true?

(a) $192 + n = 168$ (None)

(b) $192 + n = 268$ ($n = 76$)

(c) $312 - n = 314$ (None)

(d) $312 - n = 310$ ($n = 2$)

(e) $n - 12 = 26$ ($n = 38$)

(f) $12 - n = 26$ (None)

(g) $26 - 21 = n$ ($n = 5$)

(h) $21 - 26 = n$ (None)

7. BRAINTWISTER.

(a) The two numbers you operate on are n and n. The operation you use is addition. The result is 200. What number is n? ($n + n = 200$, $n = 100$)

(b) Follow the directions **of exercise (a) but replace** 200 with 582. ($n + n = 582$, $n = 291$)

8. BRAINTWISTER. What is wrong with this problem? The two numbers you operate on are n and n. The operation you use is subtraction. The result is 10. What number is n?
(If the statement were true, we would have $n - n = 10$. We know the sentence is false because $n - n = 0$ or $n - n \neq 10$.)

9. BRAINTWISTER. Two numbers operated on are n and 376. The result is 593. Write two true mathematical sentences using n, 376, and 593. In each mathematical sentence n will be a different number. ($n + 376 = 593$, $n = 217$
$n - 376 = 593$, $n = 969$)

10. BRAINTWISTER. Two numbers operated on are n and 376. The result is 89. Can you write one or two true mathematical sentences using n, 376, and 89? Why?
($n - 376 = 89$, $n = 465$
$376 - n = 89$, $n = 287$
For $n + 376 = 89$, there is no whole number for n because
$n = 89 - 376$
Subtraction is not always possible within the set of whole numbers.)

Exercise Set 18

1. A model plane costs $2.15. Joe had some money and then he earned $1.58. Then he had exactly enough to buy the plane. How much did he have before earning $1.58?
(n+158 = 215, or 215-158 = n. Joe had 57¢ before earning $1.58.)

2. The fourth grade class collected 287 more pounds of old newspapers than the fifth grade class. The fourth grade class collected 512 pounds. How much did the fifth grade collect? (512 - 287 = n, or n + 287 = 512. The fifth grade class collected 225 pounds of old newspapers.)

3. 721 is the largest 3 digit number that can be written using each of the digits 7, 2, and 1. What is the smallest number that can be so written? (127) What must be added to the smaller number to get the larger? (127+n=721, or 721-127=n

4. Mary went to the store to buy one loaf of bread and one dozen eggs. Bread is 29¢ a loaf and eggs 65¢ a dozen. Using only the above information which of these questions can you answer?
   (a) What is the cost of Mary's purchases? (29+65=n. The cost is 94 cents.)
   (b) How much in all did Mary pay for bread? (29¢)
   (c) How much change did she bring home? (We do not know how much money she had.)
   (d) If she gave the clerk a $5 bill, how much change did she receive? (94+n = 500, or 500-94=n. She received 406 cents or $4.06 in change.)

5. The East School had a newspaper and magazine drive. Room A collected 1,546 pounds, Room B collected 2,875 pounds, and Room C collected 5,324 pounds. How many pounds of paper did these three rooms collect in all?
($1546 + 2875 + 5324 = n$. the three rooms collected 9,745 pounds.)

6. BRAINTWISTER. Use the numbers 2, 3, 4, 5, 6, 7, 8, 9 and 10 to make a magic square. Hint: the sum of each row, column, and diagonal is 18.

(

5	4	9
10	6	2
3	8	7

or

9	2	7
4	6	8
5	10	3

)

7. BRAINTWISTER. (a) What number is n if
$(6 - n) + 4 = (6 + n) - 4$ ($n = 4$)
(b) How many counting numbers are there between 194 and 275? (80)

8. BRAINTWISTER. Each mathematical sentence below is true. In which is n not a whole number? (d)
(a) $n - n = n$ ($n = 0$)
(b) $10 - n = n$ ($n = 5$)
(c) $(3 + 2) + 2 = n$ ($n = 7$)
(d) $(3 + 2) + n = 2$ (not a whole number)

9. BRAINTWISTER. Find n so each mathematical sentence is true.
(a) n is less than 2. ($n = 0$ or $n = 1$)
(b) n is less than 8 and n is more than 6. ($n = 7$)
(c) n added to 3 is less than 5. ($n = 0$ or $n = 1$)
(d) $n < 12$ and $n > 10$ ($n = 11$)
(e) $n + 4 < 6$ ($n = 0$ or $n = 1$)

Exercise Set 19

At Jordon school, the cafeteria served lunch to:

195	children on Monday
218	children on Tuesday
198	children on Wednesday
203	children on Thursday
194	children on Friday

Use the above information to solve problems 1-8.

1. How many children were served lunch during the week?
(195+218+198+203+194=n. There were 1008 children served lunch during the week.)

2. How many more than 1,000 lunches were served during the week? (1008-1000=n, or 1000+n=1008. There were 8 lunches more than 1000 served during the week.)

3. Find the two days on which the most lunches were served. The total number of lunches for these two days was how many less than 500? (500 -(218+203) = n. The total number of lunches for these two days was 79 less than 500.)

4. The total number of lunches served the first three days of the week is how many more than the number served the last two days of the week? ((195+218+198) - (203+194) = n. There were 214 more lunches served the first three days of the week than were served the last two days of the week.)

The mathematical sentences in exercises 5 through 8 answer what questions about the number of lunches served?

5. $195 + n = 218$
(How many more lunches were served on Tuesday than were served on Monday?)

6. $n = 198 + 194$
(How many lunches were served on Wednesday and Friday?)

7. $(198 + 203) + 194 = n$
(What was the total number of lunches served the last three days of the week?)

8. $218 - 203 = n$
(How many more lunches were served on Tuesday than were served on Thursday?)

The prices of some card games are: Old Maid 26¢, Hearts 19¢, Play Your Hunch 17¢, and Rummy 24¢.

Which of the mathematical sentences in the box can be used to answer exercises 9 through 11?

(a) $100 + n = 17 + 19$
(b) $(19 + 24) - 26 = n$
(c) $(24 + 19) + 26 = n$
(d) $n - (19 + 24) = 26$
(e) $100 - (17 + 19) = n$
(f) $(24 + 19) - n = 26$

9. What is the total cost of Rummy, Hearts, and Old Maid? (c)

10. How much change do you receive from $1.00 if you buy Play Your Hunch and Hearts? (e)

11. How much more will it cost to buy the 2 games Hearts and Rummy than 1 game of Old Maid? (b) or (f)

The mathematical sentences in exercises 12 through 17 answer what questions about the cost of the card games?

12. $17 + 24 = n$
(What is the total cost of Play Your Hunch and Rummy?)

13. $n = 19 - 17$
(How much more does Hearts cost than Play Your Hunch?)

14. $24 - n = 17$
(How much more is the cost of Rummy than Play Your Hunch?)

15. $n + 26 = 19$
(There is no whole number n that will make the statement true.)

16. $n = 17 + 26$
(What is the total cost of Play Your Hunch and Old Maid?)

17. $17 + n = 24$
(How much more is the cost of Rummy than Play Your Hunch?)

ENRICHMENT

Objective: To help pupils review and extend their understanding of union and intersection of sets. To help them comprehend the meaning of an operation on numbers

Vocabulary: The words circ, bow, wob, star, beta, pick and alpha were invented for use in **Exercise Set 20.** It is not intended that they become part of the pupils' vocabulary.

Exploration:

The teacher should study Exercise Sets 20 and 21 carefully and decide which of his pupils can profit from the study of this enrichment material.

The teacher may refer to Teachers' Commentary, **Chapter 3 for backgrouhd and suggestions on teaching union of sets, Exercise Set 21.**

Because pupils are familiar with addition and subtraction they often do not sense the significance of the statement that they are operations. Make-believe operations are introduced in Exercise Set 20 to help them comprehend what is meant by operation.

This set of exercises is written so that pupils have the opportunity to discover the rule for the make-believe operations. Try to make a game out of this lesson. The pupils may be called discoverers or inventors. The **Pupils' Book** may be used as a basis for the study of this topic or the teacher may wish to use the following as an introduction before turning to the book.

Today we are going to do some inventing. We will invent some make-believe operations and some symbols to indicate those operations. First, name **the operations of mathematics that you know. (Addition, subtraction, multiplication, and division) Write the symbol for each of these on the board.**

The new make-believe operation I have invented is indicated by the symbol, ▭ . Help me invent a name for it. (Pupils may **think of "box" or "rectangle." Choose a strange name such as "rect.")**

"Rect" is an operation on two numbers. The result of the operation, rect is found by this rule, "add 2 to the sum of the numbers." So $4 \square 3 = 9$ and $1 \square 5 = 8$. Tell me the result of $6 \square 3$; of $2 \square 3$; of $6 \square 7$, etc. (11, 7, 15)

> Urge pupils to make up a symbol and a rule represented by the symbol.

This is another symbol to indicate an operation. (Write $\sim$ on the board.) Its name is "Nac." It is an operation on two numbers. Here are some mathematical sentences using $\sim$:

$3 \sim 1 = 3$ $5 \sim 7 = 11$ $4 \sim 9 = 12$

How many of you can find $2 \sim 4$?

> Write more statements such as $1 \sim 1 =$ ___ $6 \sim 2 =$ ___ on the board. Let pupils who have discovered the rule supply answers. (The symbol $\sim$ means, "subtract 1 from the sum of the two numbers.") After a number of pupils have discovered the rule let one state it in words.
>
> There are other examples, like those described above, in the pupils' book.
>
> If the teacher decides that only a few pupils should study Exercise Set 20 they may do so independently.

ENRICHMENT

Exercise Set 20

Meaning of Operation

You have been studying addition and subtraction, two of the operations of mathematics. They are operations on two numbers. The symbols that indicate these operations are + and -. Now we are going to "make up" some operations. They are "make-believe" operations and are not found in mathematics books. They have been invented to see if you can discover their meaning.

1. One make-believe operation is named "circ." The symbol to indicate circ is $\odot$. $2 \odot 4$ is read, "Two circ four." Circ means add 3 to the first number and then subtract the second number from that sum. Thus $2 \odot 4 = 1$. Find n for each of these.

   (a) $3 \odot 2 = n$ (n=4)
   (b) $8 \odot 1 = n$ (n=10)
   (c) $5 \odot 1 = n$ (n=7)
   (d) $4 \odot 4 = n$ (n=3)
   (e) $7 \odot 6 = n$ (n=4)
   (f) $9 \odot 2 = n$ (n=10)

2. Another make-believe operation is named, "bow." The symbol to indicate bow is T . 3 T 4 is read, "Three bow four." Bow means choose the smaller number. Thus

   8 T 5 = 5. Find n for each of these.

   (a) 2 T 3 = n (n=2)
   (b) 12 T 8 = n (n=8)
   (c) 4 T 1 = n (n=1)
   (d) 9 T 10 = n (n=9)
   (e) 8 T 3 = n (n=3)
   (f) 7 T 6 = n (n=6)

3. Another operation is named, "wob." The symbol to indicate wob is $\perp$. $3 \perp 4$ is read "Three wob four." Here are some results of the operation, wob, on two numbers. Try to find the meaning of wob. (Wob means choose the larger number.)

(a) $4 \perp 6 = 6$

(b) $8 \perp 0 = 8$

(c) $8 \perp 1 = 8$

(d) $2 \perp 6 = 6$

(e) $5 \perp 9 = 9$

(f) $7 \perp 7 = 7$

Find n for each of the following:

(g) $5 \perp 6 = n$ ($n = 6$)

(h) $1 \perp 9 = n$ ($n = 9$)

(i) $7 \perp 10 = n$ ($n = 10$)

(j) $6 \perp 8 = n$ ($n = 8$)

(k) $2 \perp 0 = n$ ($n = 2$)

(l) $9 \perp 2 = n$ ($n = 9$)

4. The symbol * is to be a sign of operation. $3 * 4$ tells you to operate on 3 and 4 in a certain way. It is read, "Three star four." Here are some results of the operation, star, on two numbers. Try to find the meaning of star. (Star means add 1 to the sum.)

(a) $3 * 4 = 8$

(b) $5 * 6 = 12$

(c) $2 * 6 = 9$

(d) $3 * 7 = 11$

(e) $1 * 1 = 3$

(f) $5 * 4 = 10$

Find n for each of the following:

(g) $2 * 4 = n$ ($n = 7$)

(h) $8 * 7 = n$ ($n = 16$)

(i) $3 * 6 = n$ ($n = 10$)

(j) $5 * 9 = n$ ($n = 15$)

(k) $1 * 0 = n$ ($n = 2$)

(l) $1 * 6 = n$ ($n = 8$)

5. Another operation is called, "pick." The symbol for pick is ┘. Try to find the meaning of ┘ from these examples.
(Pick means to divide the sum by 2.)

(a) 3 ┘ 5 = 4 (c) 2 ┘ 4 = 3 (e) 7 ┘ 5 = 6

(b) 0 ┘ 2 = 1 (d) 8 ┘ 6 = 7 (f) 9 ┘ 7 = 8

6. Another operation is called, "alpha." The symbol for alpha is ℒ. It is an operation on one number. Try to find the meaning of ℒ from these examples. (Alpha means to double the number.)

(a) ℒ 3 = 6 (b) ℒ 0 = 0 (c) ℒ 5 = 10 (d) ℒ 8 = 16

What is n in each of the following?

(e) ℒ 4 = n (n=8) (f) ℒ 9 = n (n=18) (g) ℒ 1 = n (n=2) (h) ℒ 7 = n (n=14)

7. SUPER BRAINTWISTER. Another operation is called, "beta." The symbol to indicate beta is **B**. Try to find the meaning of beta from these examples: (Beta means to subtract the sum of the two numbers from 12.)

(a) 3 **B** 4 = 5 (c) 2 **B** 8 = 2 (e) 6 **B** 1 = 5

(b) 1 **B** 2 = 9 (d) 7 **B** 5 = 0 (f) 3 **B** 3 = 6

Find n for each of the following:

(g) 2 **B** 3 = n (n=7) (i) 5 **B** 6 = n (n=1) (k) 1 **B** 0 = n (n=11)

(h) 8 **B** 4 = n (n=0) (j) 4 **B** 2 = n (n=6) (l) 5 **B** 5 = n (n=2)

8. SUPER BRAINTWISTER. For which of the operations in exercises 1-7 does the commutative property seem to hold? (bow, wob, star, pick, beta)

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Exercise Set 21

UNION OF SETS

Pretend you have Set A and Set B.

Call Set C the intersection of Set A and Set B.

Call Set D the union of Set A and Set B.

Copy and fill in this table. (You may need to draw some pictures.)

	Number of members in Set A	Number of members in Set B	Number of members in Set C (Intersection)	Number of members in Set D (Union)
(1)	7	8	0	15
(2)	7	8	2	13
(3)	7	9	4	(12)
(4)	8	3	0	(11)
(5)	8	3	n	$(11-n)$
(6)	8	m	0	$(8+m)$
(7)	p	r	2	$((p+r)-2)$
(8)	p	r	5	$((p+r)-5)$

Chapter 7

TECHNIQUES OF MULTIPLICATION AND DIVISION

PURPOSE OF THE UNIT

1. To help children understand the techniques of multiplication and division (Throughout the unit the process of division includes both division without a remainder and division with a remainder)

2. To help children understand that they can multiply and divide large numbers if they know the multiplication facts and the properties of multiplication and division

3. To help children develop skill in multiplication and division and in checking the results of these processes (A high level of skill will not be expected until after the study of Chapter 3 in Grade 5)

4. To help children improve their problem solving ability through the use of mathematical sentences in situations suggesting multiplication and division

MATHEMATICAL BACKGROUND

In this unit we learn how to use the properties of multilication and division (as studied in Chapter 4) to develop techniques of multiplying and dividing whole numbers.

To do this we make use of the commutative and associative properties of multiplication, the distributive property, special properties of both 0 and 1, the decimal system of numeration, and the multiplication facts for purposes of determining the product of numbers larger than 9 and for expressing one number as a multiple of another number.

The process of multiplication. We have associated the number $a \times b$ with an array of a rows and b columns. With an array of 3 rows and 4 columns, we associate the number, 3×4. Since 3×4 is not a standard form for expressing a counting number, we can set up a correspondence between the elements of the set and the elements of standard sets, matching this particular set with one which has the number property 12. We can then name the number of the set, either as 3×4 or 12. In $3 \times 4 = 12$, we call 12 the product of 3 and 4. We call 3 and 4 factors of 12.

To find the product of 7 and 24, we can first express 24 as $20 + 4$. We then think of the decimal numeral for 7×24, by using the decimal numeral for 7×20 or $(7 \times 2) \times 10$, and the decimal numeral which expresses the same number as does (7×4). The following suggest different thought patterns for finding the number n represents in the sentence,

$$7 \times 24 = n.$$

We may first associate this number with an array as in Figure 1. (7 × 20) = 140 and 7 × 4 = 28.

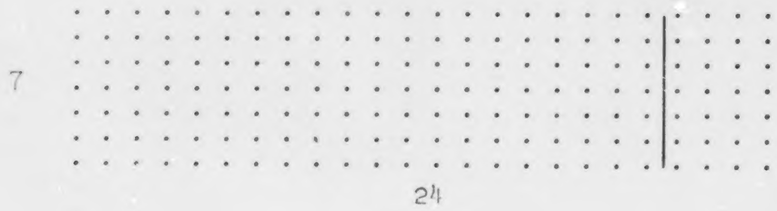


Figure 1

$$\begin{array}{r} 24 \\ \underline{\times\ 7} \end{array} \quad \text{may be written} \quad \begin{array}{ccccc} 20 & + & 4 & & \\ \hline & & \times\ 7 & & \\ 140 & + & 28 & = & 168 \end{array}$$

or

$$\begin{array}{rl} 24 & \\ \underline{\times\ 7} & \\ 140 & (7 \times 20) \text{ or } (7 \times 2) \times 10 \\ \underline{28} & (7 \times 4) \\ 168 & \end{array}$$

or

$$\begin{array}{rl} 24 & \\ \underline{\times\ 7} & \\ 28 & (7 \times 4) \\ \underline{140} & (7 \times 20) \\ 168 & \end{array}$$

or, if one can remember the "2 tens" and add it to the "14 tens", then we can write

$$\begin{array}{r} 24 \\ \underline{\times\ 7} \\ 168 \end{array} \qquad \begin{array}{l} 7 \times 4 = 20 + 8 \\ 7 \times 20 = 140 \end{array} \qquad 140 + 20 + 8 = 168$$

The form used is determined by the level of skill achieved. It would be expected that a child learning to multiply would progress from lower levels to higher levels of skill.

These procedures may be used for finding products of pairs of larger numbers. The thought process and the record of thoughts, however, become more complex.

Suppose $20 \times 34 = n$. This number, expressed as a product expression, can be associated with an array of 20 rows and 34 columns or with a collection of 20 sets of objects where the number of objects in each set is 34. We can express 20 as (2×10) and 34 as $(30 + 4)$.

Then, $20 \times 34 = 20 \times (30 + 4)$. $20 \times (30 + 4) = (20 \times 30) + (20 \times 4)$. To find the decimal numeral for (20×30) we express 20 as (2×10) and 30 as 3×10. Then $20 \times 30 = (2 \times 10) \times (3 \times 10)$. By the commutative and associative properties, we know that $2 \times 10 \times 3 \times 10 = 2 \times 3 \times 10 \times 10$. We can think of $2 \times 3 \times 10 \times 10$ as $(2 \times 3) \times (10 \times 10)$. We know the products associated with the pair of "single-digit" numbers, 2 and 3. Our system of numeration makes multiplying by 100 most convenient. These ideas are used in developing a scheme for finding the decimal numeral which names the same number as does 20×34.

Or suppose given the product expression (12×14), we wish to determine the decimal numeral that names the same number. Let us again represent an array with which this number might be associated. (See Figure 2 below.)

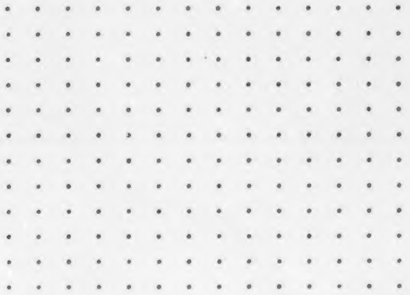


Figure 2

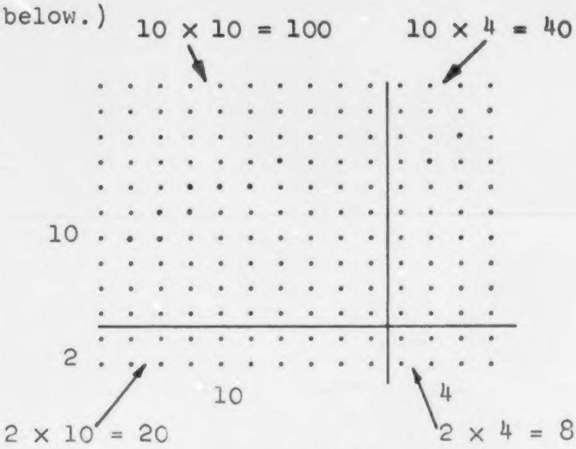


Figure 3

We can separate it into several smaller arrays, and for each array we can readily use a decimal numeral to express the number of elements in that set. (See Figure 3) That is,

$$\begin{aligned} 12 \times 14 &= (10 + 2) \times (10 + 4) \\ &= (10 \times 10) + (10 \times 4) + (2 \times 10) + (2 \times 4). \end{aligned}$$

When symbols alone are used to express our thoughts, the record may be expected to move from a procedure closely resembling this to the brief algorism that many of us know. For example, to find the number n represents in the mathematical sentence $12 \times 14 = n$, we may write the following:

$$\begin{array}{rl} 14 & \\ \underline{\times\ 12} & \\ 8 & (2 \times 4) \\ 20 & (2 \times 10) \\ 40 & (10 \times 4) \\ \underline{100} & (10 \times 10) \\ 168 & \end{array}$$

This resembles Figure 3. Too, order can vary as displayed at the right.

At this stage it is not important to establish a particular order for all pupils.

$$\begin{array}{rrl} 14 & 14 & \\ \underline{\times\ 12} & \underline{\times\ 12} & \\ 100 & 20 & \\ 40 & 8 & \text{etc.} \\ 20 & 100 & \\ \underline{8} & \underline{40} & \\ 168 & 168 & \end{array}$$

Later, $\begin{array}{r} 14 \\ \underline{\times\ 12} \\ 28 \\ \underline{140} \\ 168 \end{array}$ or $\begin{array}{r} 14 \\ \underline{\times\ 12} \\ 140 \\ \underline{28} \\ 168 \end{array}$ and still later $\begin{array}{r} 14 \\ \underline{\times\ 12} \\ 28 \\ \underline{14} \\ 168 \end{array}$ and still later, and for some $\begin{array}{r} 14 \\ \underline{\times\ 12} \\ 168 \end{array}$

The process of division. In developing a technique for dividing one number by another, we make use of what we might call a "one-sided" distributive property for division. For example, to divide 165 by 15, we express 165 as a sum of multiples of 15, i.e., $165 = 60 + 60 + 45$. Then

$$(60 + 60 + 45) \div 15 = (60 \div 15) + (60 \div 15) + (45 \div 15)$$

Or, $$= (4 + 4 + 3) = 11.$$

Hence, $$165 \div 15 = 11.$$

For pairs of numbers, where one is not a factor of the other, we find the largest multiple of one number which is less than the other. In order to divide 137 by 14, it is possible to express 137 as the sum of multiples of 14 and a final addend less than 14. The use of the distributive property is not appropriate since all addends are not multiples of 14. So, we use the following:

$$137 = 70 + 56 + 11$$

Then, $137 = \left[(70 \div 14) + (56 \div 14)\right] \times 14 + 11$, or more simply written, $137 = (5 + 4) \times 14 + 11$

Thus, when we divide 137 by 14, the quotient is 9 and the remainder is 11. The largest multiple of 14 less than 137 is $(137 - 11)$ or 126.

For pairs of numbers such as these, where one is not a factor of the other, the mathematical sentence in the form of $\underline{c \div b = a}$ or $\underline{c = a \times b}$ does not apply. Instead, we use the sentence of the form $c = (a \times b) + r$. Of course, we see if r is zero, then the second takes a form of the first. Observe that $r < b$.

This is the basis for developing a computational scheme, whereby we subtract multiples of one number from the other in order to determine the largest multiple of one number that is less than the other.

There are different ways in which the thought processes may be recorded. Here are some ways that will be used throughout this unit.

$$\begin{aligned} p &= 84 \div 4 \\ &= (80 + 4) \div 4 \\ &= (80 \div 4) + (4 \div 4) \\ &= 20 + 1 \\ &= 21 \end{aligned}$$

Or, using smaller multiples of 4, either of the forms, A or B, shown below may be used.

A.

$$\begin{array}{r|rl}
4\,\overline{)\,84} & & \\
\underline{-\ 40} & 10 & (10 \times 4) \\
44 & & \\
\underline{-\ 40} & 10 & (10 \times 4) \\
4 & & \\
4 & 1 & (1 \times 4) \\
\hline
0 & 21 &
\end{array}$$

or

$$\begin{array}{r|rl}
4\,\overline{)\,84} & & \\
\underline{-\ 80} & 20 & (20 \times 4) \\
4 & & \\
-\ 4 & 1 & (1 \times 4) \\
\hline
0 & 21 &
\end{array}$$

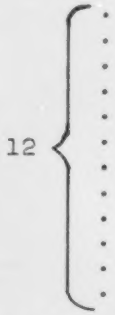
B.

$$\begin{array}{rl}
\underline{21} & \\
1 & \\
10 & \\
10 & \\
4\,\overline{)\,84} & \\
\underline{40} & (10 \times 4) \\
44 & \\
\underline{40} & (10 \times 4) \\
4 & \\
\underline{4} & (1 \times 4) \\
0 &
\end{array}$$

or

$$\begin{array}{rl}
\underline{21} & \\
1 & \\
20 & \\
4\,\overline{)\,84} & \\
\underline{80} & (20 \times 4) \\
4 & \\
\underline{4} & (1 \times 4) \\
0 &
\end{array}$$

In developing a technique for dividing a number c by a number a, we can consider the separation of a set of c elements (dots, for example) into a rows with the same number of dots in each row. For example, in dividing 132 by 12 we can make an array of 12 rows, beginning as indicated.



Then we form other columns of dots until the total count is 132. The total number of columns is 11. We know then that $12 \times 11 = 132$ and consequently $132 \div 12 = 11$. Likewise, we could have made an array of 12 columns instead of 12 rows. But it may be desirable to confine the pupils' attention to just one array in which the number of rows is the number given as the known factor, in this case 12. The finding of the unknown factor, 11, accomplishes the dividing of 132 by 12.

If we were required to divide 135 by 12, we know there is no whole number for n so that $12 \times n = 135$. (In an array this would be shown by 12 rows of 11 dots each and 3 dots remaining for a partial twelfth column.) In this case, we cannot divide two whole numbers and get a third whole number. Since it is not always possible to find a whole number for an unknown factor if the product and the known factor are whole numbers, division is not always possible within the set of whole numbers. We cannot write $135 = 12 \times n$ where n represents a whole number. But expressions such as "135 divided by 12" may be interpreted in relation to the partitioning of a set of 135 objects into the largest possible number of equivalent subsets with 12 objects in each subset with a remaining subset of fewer than 12 objects. This is described by a mathematical sentence of the form

$$135 = (n \times 12) + r$$

in which n and r are whole numbers and n is as large as possible and $r < 12$. In this example $n = 11$ and $r = 3$:

$$135 = (11 \times 12) + 3$$

Since the operation of division is an operation with two numbers (dividend and divisor) to yield one number (quotient) the process of finding the two numbers n and r is not the operation of division. The algorism for recording one's thoughts in determining 11 and 3 in the above example may be the same as the algorism for dividing 132 by 12 but there is an essential difference in recording the final results.

$$\begin{array}{r@{}r|r} 12) & \overline{135} & \\ & \underline{120} & 10 \\ & 15 & \\ & 12 & 1 \\ \hline & 3 & 11 \end{array} \qquad\qquad \begin{array}{r@{}r|r} 12) & \overline{132} & \\ & \underline{120} & 10 \\ & 12 & \\ & 12 & 1 \\ \hline & & 11 \end{array}$$

Some distinction needs to be made in the language used in "dividing 132 by 12" and "dividing 135 by 12."

In "dividing 132 by 12" we shall use the following terminology: 132 is the product of the known factor 12 and an unknown factor represented by n. But in "dividing 135 by 12" we shall use the terminology: Finding quotient and remainder. It is true, of course, that prior to determining if there is a whole number for the unknown factor, the pupils cannot be aware of which of the two situations exists, i.e., whether they are finding an unknown factor or finding a quotient and remainder. The larger portion of the material in division presents the "product and unknown factor" situations first and the "quotients with remainders" are introduced in the last few sections.

Whenever we try to think about a given product and a known factor of zero, we have difficulty finding the other factor. Suppose $24 \div 0 = n$. Then, $24 = 0 \times n$. Clearly there is no whole number n we can use since the product of 0 and any whole number is again zero. What can be said about $0 \div 0$? If $0 \div 0 = n$, then $0 \times n = 0$. Here we have a somewhat different situation since any whole number can serve as the other factor. In brief, when one factor is 0, either we cannot determine the other factor or any whole number will do. Consequently, we usually agree to avoid division by 0. This is no real hardship since we do not meet physical situations which require such sentences for their description.

TEACHING THE UNIT

REVIEWING MULTIPLICATION AND DIVISION

Objective: To help children learn techniques of multiplication and how these techniques depend upon the basic properties of multiplication

> No lengthy discussion of multiplication is required. Arrays are used to help develop multiplication algorisms. Because of its continual use and great importance, the distributive property should be reviewed.

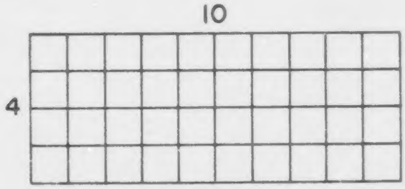
Materials: Arrays

Vocabulary: Partial products, vertical

Exploration:

Multiplication is a mathematical operation on two numbers to obtain a third number. Give some products of pairs of numbers you know. Do you know how to find the product of any two numbers you can think of? Can you give some examples of numbers you don't know how to multiply? (These might be such as 325×78, 24×96, etc.)

We know the products of pairs of numbers less than ten. We now want to learn more about finding products of pairs of numbers greater than ten. Let us begin by remembering some simple products. Suppose we think of $n = 4 \times 10$ as the number of elements in an array of 4 rows and 10 columns.



If we do not know the number n represents, we can find it by thinking of ways of separating the array. How may we separate this array into two arrays?

Have children suggest several possibilities, making certain that they include two 4 by 5 arrays.

Bring out the idea that it is very easy to find the product of two numbers when one factor is 10 and the other is less than 10. We can think of 4×10 as 40 (4 tens), 5×10 as 50 (5 tens), etc. Also, $10 \times 4 = 40$ or 4 tens, etc.

Do you think you know a way to find the product of two numbers when one is 10 and the other is greater than 10? (Try some examples.) What is the product of 10 and 10? of 11 and 10? etc.

Use such examples as:

$$\begin{aligned}10 \times 15 &= 10 \times (10 + 5)\\ &= (10 \times 10) + (10 \times 5)\\ &= 100 + 50\\ &= 150\end{aligned}$$

$$\begin{aligned}11 \times 10 &= (10 + 1) \times 10\\ &= (10 \times 10) + (1 \times 10)\\ &= 100 + 10\\ &= 110\end{aligned}$$

$$\begin{aligned}10 \times 64 &= 10 \times (60 + 4)\\ &= (10 \times 60) + (10 \times 4)\\ &= 600 + 40\\ &= 640\end{aligned}$$

Lead children to use 10 as a factor as often as possible. Work other problems as a class activity before Exercise Set 2. Exercises 19 and 20 are given as a "lead-in" to new material.

Chapter 7

TECHNIQUES OF MULTIPLICATION AND DIVISION

OPERATIONS

We think of addition, subtraction, multiplication, and division as the four basic operations of arithmetic.

We have learned that an operation on numbers is a way of thinking about two numbers and getting one number as a result.

When we think about 12 and 3 and get 15, we are adding. When we think about 12 and 3 and get 9, we are subtracting. When we think about 12 and 3 and get 36, we are multiplying. When we think about 12 and 3 and get 4, we are dividing.

MULTIPLICATION

We express multiplication like this:

$$9 \times 4 = 36.$$

We read the sentence like this:

9 times 4 is equal to 36.

9 times 4 equals 36.

We know that:

9 is a factor of 36.

4 is a factor of 36.

36 is the product of 9 and 4.

DIVISION

We express division like this:

$$36 \div 9 = n$$

$$36 = n \times 9$$

or

$$36 = 9 \times n.$$

We read the sentence like this:

36 divided by 9 is equal to n.

36 is equal to what times 9?

36 is equal to 9 times what number?

We know that:

36 is the product of 9 and n.

9 is a known factor of 36.

n is an unknown factor of 36.

THE DISTRIBUTIVE PROPERTY OF MULTIPLICATION OVER ADDITION

Methods of multiplication depend on expressing one factor as a sum and then using the distributive property of multiplication over addition.

To multiply 48×6, you can think of 48 as $(40 + 8)$. Then we multiply each number by the factor 6. We use the distributive property.

$6 \times 48 = 6 \times (40 + 8)$	Rename 48 as $(40 + 8)$.
$= (6 \times 40) + (6 \times 8)$	Distribute the 6 over $(40 + 8)$.
$= 240 + 48$	The product of 6 and 40 is 240. The product of 6 and 8 is 48.
$= 288$	The sum of 240 and 48 is 288.

THE DISTRIBUTIVE PROPERTY OF DIVISION OVER ADDITION

To divide 75 by 5, we express 75 as $(50 + 25)$. Then we divide both numbers by 5.

$75 \div 5 = (50 + 25) \div 5$	Rename 75 as $(50 + 25)$.
$= (50 \div 5) + (25 \div 5)$	Distribute the 5 over $(50 + 25)$.
$= 10 + 5$	Divide 50 by 5. Divide 25 by 5.
$= 15$	The sum of 10 and 5 is 15.

Exercise Set 1

Write the numerals from 1 to 20 on your paper. If a statement is true, write true. If a statement is false, write false.

1. Using the set of whole numbers, you can multiply any pair of numbers and always get a whole number for their product. (true)
2. Using the set of whole numbers, you can divide any pair of numbers and always get a whole number for the unknown factor. (false)
3. $273 \times 846 = 846 \times 273$ (true)
4. $3 \div 1 = 1$ (false)
5. $69 \div 3 = 3 \div 69$ (false)
6. $17 = 17 \div 1$ (true)
7. $6 \times 0 = 6$ (false)
8. $1 \times 9 = 9$ (true)
9. $0 \div 6 = 0$ (true)
10. $6 \times 9 < 7 \times 9$ (true)
11. $58 \times 69 > 69 \times 58$ (false)
12. $48 \div 4 = (40 \div 4) + (8 \div 4)$ (true)
13. $(20 + 4) \times 7 = (20 \times 7) + (4 \times 7)$ (true)
14. $2 \times (3 \times 17) = (3 \times 17) \times 2$ (true)
15. $2 \times 34 = (2 \times 30) + (2 \times 40)$ (false)
16. $(21 \times 7) \div 7 = (21 \div 7) \times 7$ (true)
17. $(48 \div 6) \div 2 = 48 \div (6 \div 2)$ (false)
18. $(5 \times 30) + (5 \times 6) = 5 \times 36$ (true)
19. $(47 \times 18) + (47 \times 12) = 47 \times 30$ (true)
20. $(12 \div 3) + (12 \div 3) = 24 \div 3$ (true)

Exercise Set 2

Find a decimal numeral for n in each sentence.

1. $10 \times 18 = n$ $(n = 180)$
2. $17 \times 10 = n$ $(n = 170)$
3. $27 \times 10 = n$ $(n = 270)$
4. $10 \times 35 = n$ $(n = 350)$
5. $10 \times 107 = n$ $(n = 1070)$
6. $12 \times 10 = n$ $(n = 120)$
7. $120 \times 10 = n$ $(n = 1200)$
8. $10 \times 19 = n$ $(n = 190)$
9. $30 \times 100 = n$ $(n = 3000)$
10. $300 \times 10 = n$ $(n = 3000)$
11. $10 \times 47 = n$ $(n = 470)$
12. $89 \times 10 = n$ $(n = 890)$
13. $54 \times 10 = n$ $(n = 540)$
14. $10 \times 98 = n$ $(n = 980)$
15. $10 \times 125 = n$ $(n = 1250)$
16. $314 \times 10 = n$ $(n = 3140)$
17. $412 \times 10 = n$ $(n = 4120)$
18. $842 \times 10 = n$ $(n = 8420)$

BRAINTWISTERS:

19. $17 \times 20 = n$

$$\left(\begin{aligned} 17 \times 20 &= 17 \times (10 + 10) \\ &= (17 \times 10) + (17 \times 10) \\ &= 170 + 170 \\ &= 340 \end{aligned}\right)$$

20. $12 \times 30 = n$

$$\left(\begin{aligned} 12 \times 30 &= 12 \times (3 \times 10) \\ &= (12 \times 3) \times 10 \\ &= 36 \times 10 \\ &= 360 \end{aligned} \quad \underline{\underline{\text{OR}}} \quad \begin{aligned} 12 \times 30 &= 12 \times (10 + 10 + 10) \\ &= (12 \times 10) + (12 \times 10) + (12 \times 10) \\ &= 120 + 120 + 120 \\ &= 360 \end{aligned}\right)$$

MULTIPLYING BY MULTIPLES OF TEN

Objective: To learn how to multiply two numbers when one number is a multiple of 10, that is, 20, 30, 40, etc.

Vocabulary: Multiples

Exploration:

We can use what we know about 10 as a factor to learn how to use 20 as a factor. How may we think of the factor 20 in order to use what we already know? ($20 = 10 + 10$ or 2×10) Suppose we try some examples:

$7 \times 20 = 7 \times (10 + 10)$	Rename 20 as $(10 + 10)$.
$= (7 \times 10) + (7 \times 10)$	Distribute 7 over $(10 + 10)$.
$= 70 + 70$	Multiply 7 and 10.
$= 140$	Add 70 and 70.

or

$7 \times 20 = 7 \times (2 \times 10)$	Rename 20 as (2×10).
$= (7 \times 2) \times 10$	Use the associative property.
$= 14 \times 10$	Multiply 7 and 2.
$= 140$	Multiply 14 and 10.

If we use the second way, we will not have to add large numbers.

$$\begin{aligned} 9 \times 20 &= 9 \times (2 \times 10) \\ &= (9 \times 2) \times 10 \\ &= 18 \times 10 \\ &= 180 \end{aligned}$$

To use 20 as a factor, what do we need to know? (We need to know how to use 2 and 10 as factors.) Can you use these same ideas if 30, 40, ... 90 are factors? (Yes, if you know facts for 3's, 4's, ... 9's.) Let's try one of these.

$8 \times 70 = 8 \times (7 \times 10)$	Rename 70 as 7×10.
$= (8 \times 7) \times 10$	Use the associative property.
$= 56 \times 10$	Multiply 8 and 7.
$= 560$	Multiply 56 and 10.

> It will be very helpful for a pupil to be able to write $8 \times 70 = 560$ without having to use all of the steps shown in the Exploration. Each pupil should know how to use the basic properties to find such products.
>
> It may be necessary to think about solving problems using multiplication before working the problems in Exercise Set 4.

MULTIPLYING BY MULTIPLES OF TEN

We have learned how to multiply two numbers when one of the numbers is 10.

Now we want to learn to multiply two numbers when one of the numbers is a multiple of 10. Multiples of 10 are 10, 20, 30, 40, 50, and so on. Can you name some other multiples of 10?

Suppose we find the product of 7 and 20. To multiply 7 and 20, we can think of 20 as (10 + 10). Then,

$7 \times 20 = 7 \times (10 + 10)$ Rename 20 as (10 + 10).
$= (7 \times 10) + (7 \times 10)$ Distribute 7 over (10 + 10).
$= 70 + 70$ Multiply 7 and 10.
$= 140$ Add 70 and 70.

Too, we can think of 20 as (2 × 10). Then,

$7 \times 20 = 7 \times (2 \times 10)$ Rename 20 as (2 × 10).
$= (7 \times 2) \times 10$ Use the associative property.
$= 14 \times 10$ Multiply 7 and 2.
$= 140$ Multiply 14 and 10.

Is it easier to find the product of 7 and 20 by the first way or the second way? Let us find the product of another pair of numbers using the second way. One of the factors is a multiple of 10. Give reasons for each step in the following example.

$$\begin{aligned} 8 \times 40 &= 8 \times (4 \times 10) \\ &= (8 \times 4) \times 10 \\ &= 32 \times 10 \\ &= 320 \end{aligned}$$

The product of 8 and 40 is 320.

$$8 \times 40 = 320.$$

Exercise Set 3

Find the decimal numeral for each product in exercises 1 through 12. In exercises 1 through 4, write each step as in the example.

$$\begin{aligned} \text{Example: } 7 \times 30 &= 7 \times (3 \times 10) \\ &= (7 \times 3) \times 10 \\ &= 21 \times 10 \\ &= 210. \end{aligned}$$

1. 5 × 80 (= 5 × (8 × 10)
   = (5 × 8) × 10
   = 40 × 10
   = 400)
2. 70 × 7 (= (7 × 10) × 7
   = (7 × 7) × 10
   = 49 × 10
   = 490)
3. 50 × 8 (= (5 × 10) × 8
   = (5 × 8) × 10
   = 40 × 10
   = 400)
4. 6 × 30 (= 6 × (3 × 10)
   = (6 × 3) × 10
   = 18 × 10
   = 180)

In exercises 5 through 12, find the decimal numeral for each product without writing all the steps. In exercise 5, can you think that 7 multiplied by 4 is 28 and that 28 multiplied by 10 is 280?

5. 7 × 40 (280)
6. 5 × 60 (300)
7. 6 × 70 (420)
8. 7 × 80 (560)
9. 60 × 9 (540)
10. 3 × 600 (1800)
11. 30 × 8 (240)
12. 9 × 90 (810)

Exercise Set 4

Use mathematical sentences to help solve each of the following problems. Express each answer in a complete sentence.

1. The beads on an abacus may be arranged so that there are 20 beads on each of 4 wires. How many beads are on this abacus?

   $20 \times 4 = n$
   $(10 \times 2) \times 4 = n$
   $10 \times (2 \times 4) = n$
   $10 \times 8 = n$
   $n = 80$

   There are 80 beads on the abacus.

2. Some land will be divided into 7 blocks. 40 houses will be built on each block. How many houses will there be on the land?

   $n = 7 \times 40$
   $= 7 \times (4 \times 10)$
   $= (7 \times 4) \times 10$
   $= 28 \times 10 = 280$

   There will be 280 houses on the land.

3. In one section of a plane there were 20 rows of seats with 5 seats in each row. How many seats were there in this section of the plane?

   $n = 20 \times 5$
   $= 100$

   There were 100 seats in this section of the plane.

4. At an assembly the chairs were arranged in 30 rows. There were 10 chairs in each row. How many chairs were set up for the assembly?

   $n = 30 \times 10$
   $= 300$

   There were 300 chairs set up for the assembly.

5. Bob bought 3 season tickets to the basketball games. Each ticket costs $3.20. How much did Bob spend for the tickets?

   $n = 3 \times 320$
   $= 9.60$

   Bob spent $9.60 for tickets.

6. On the family room floor there were 660 tiles. 304 tiles were used on the kitchen floor. How many more tiles were used on the floor of the family room than on the floor of the kitchen?

   $n = 660 - 304$
   $= 356$

   There were 356 more tiles on the floor of the family room.

MULTIPLYING BY MULTIPLES OF ONE HUNDRED

Objective: To learn how to multiply two whole numbers when one is 100 or a multiple of 100

Exploration:

We have learned how to multiply two numbers when one number is 10. We used this to find how to multiply whole numbers not greater than 10 by 20, 30, ..., 90. Now we want to learn more about multiplying by 100 and multiples of 100.

> Review with pupils such examples as 2×100, 4×100, 8×100, 100×6, etc. Then ask if they can suggest how to find the product of 4 and 200.
>
> $$\begin{aligned} 4 \times 200 &= 4 \times (2 \times 100) \\ &= (4 \times 2) \times 100 \\ &= 8 \times 100 \\ &= 800 \end{aligned}$$
>
> $$\begin{aligned} 8 \times 100 &= 8 \times (10 \times 10) \\ &= (8 \times 10) \times 10 \\ &= 80 \times 10 \\ &= 800 \end{aligned}$$

Can you see a way of multiplying by 300, 400, ..., 900? Perhaps we should use one more example.

$6 \times 700 = 6 \times (7 \times 100)$ Rename 700 as (7×100).

$= (6 \times 7) \times 100$ Use the associative property.

$= 42 \times 100$ Multiply 6 and 7.

$= 4200$ Multiply 42 and 100.

One example of the type 70×60 may be worthwhile. The multiplication of two numbers, each less than 100, may lead to multiplication where one of the factors is a number less than 10.

$$\begin{aligned}
70 \times 60 &= (7 \times 10) \times (6 \times 10) \\
&= [(7 \times 10) \times 6] \times 10 && \text{Associative Property} \\
&= [7 \times (10 \times 6)] \times 10 && \text{Associative Property} \\
&= [7 \times (6 \times 10)] \times 10 && \text{Commutative Property} \\
&= [(7 \times 6) \times 10] \times 10 && \text{Associative Property} \\
&= [42 \times 10] \times 10 && \text{Using } 42 = 7 \times 6 \\
&= 42 \times (10 \times 10) && \text{Associative Property} \\
&= 42 \times 100 \\
&= 4200
\end{aligned}$$

This rather long development should not be carried out with the pupils. They will recognize (and without being in error) that 70×60 becomes 42×100 and full details of why it is correct is not a major issue at this stage.

Children may suggest this way:

$$\begin{aligned}
70 \times 60 &= (7 \times 10) \times (6 \times 10) \\
&= 7 \times 6 \times 10 \times 10 \\
&= (7 \times 6) \times (10 \times 10) \\
&= 42 \times 100 \\
&= 4200
\end{aligned}$$

or

$$\begin{aligned}
70 \times 60 &= (7 \times 10) \times 60 \\
&= 7 \times (10 \times 60) \\
&= 7 \times 600 \\
&= 7 \times (6 \times 100) \\
&= (7 \times 6) \times 100 \\
&= 42 \times 100 \\
&= 4200.
\end{aligned}$$

MULTIPLYING BY MULTIPLES OF ONE HUNDRED

We have learned how to multiply two whole numbers when one of the numbers is 10. We have learned how to multiply two whole numbers when one is a multiple of 10. What are some multiples of 10?

Look at this example for finding the product of 6 and a multiple of 10 (30).

$$\begin{aligned} 6 \times 30 &= 6 \times (3 \times 10) \\ &= (6 \times 3) \times 10 \\ &= 18 \times 10 \\ &= 180 \end{aligned}$$

We now want to learn to multiply two whole numbers when one of the numbers is 100. We also want to learn how to find the product of two numbers when one factor is a multiple of 100. What are multiples of 100? Is 200 a multiple of 100? (yes) is 300? (yes) is 400? (yes) Can you name some other multiples of 100?
(Yes. Some examples are 500, 600, 700, 800, 900, 1000, 1100, etc.)

See if you can understand these examples.

Example 1:
$$\begin{aligned} 6 \times 100 &= 6 \times (10 \times 10) \\ &= (6 \times 10) \times 10 \\ &= 60 \times 10 \\ &= 600 \end{aligned}$$

Example 2:
$$\begin{aligned} 18 \times 100 &= 18 \times (10 \times 10) \\ &= (18 \times 10) \times 10 \\ &= 180 \times 10 \\ &= 1800 \end{aligned}$$

Example 3:
$$\begin{aligned} 6 \times 300 &= 6 \times (3 \times 100) \\ &= (6 \times 3) \times 100 \\ &= 18 \times 100 \\ &= 1800 \end{aligned}$$

Example 4:
$$\begin{aligned} 50 \times 30 &= (5 \times 10) \times (3 \times 10) \\ &= (5 \times 3) \times (10 \times 10) \\ &= 15 \times 100 \\ &= 1500 \end{aligned}$$

Example 5:
$$\begin{aligned} 16 \times 200 &= 16 \times (2 \times 100) \\ &= (16 \times 2) \times 100 \\ &= 32 \times 100 \\ &= 3200 \end{aligned}$$

Example 6:
$$\begin{aligned} 4 \times 2000 &= 4 \times (20 \times 100) \\ &= (4 \times 20) \times 100 \\ &= 80 \times 100 \\ &= 8000 \end{aligned}$$

Can you name the product just by looking at the two numbers to be multiplied? Try these.

87 × 10 (870)

5 × 60 (300)

40 × 30 (1200)

4 × 100 (400)

200 × 3 (600)

12 × 400 (4800)

How many could you do?

Now you can use what you have learned about multiplying by 10 and 100 and their multiples.

Exercise Set 5

Copy and complete each of the following.

1. 7 × 10 = (70)
2. 5 × 500 = (2500)
3. (1800) = 3 × 600
4. 100 × 8 = (800)
5. (90) = 9 × 10
6. 10 × 7 = (70)
7. 500 × 5 = (2500)
8. 600 × 3 = (1800)
9. 8 × 60 = (480)
10. 50 × 6 = (300)
11. (3000) = 500 × 6
12. (5600) = 7 × 800
13. 400 × 3 = (1200)
14. 7 × 500 = (3500)
15. 800 × 5 = (4000)
16. 50 × 60 = (3000)
17. 60 × 90 = (5400)
18. 400 × 20 = (8000)
19. 3 × 2000 = (6000)
20. 6 × 3000 = (18,000)

MORE ABOUT MULTIPLYING

Objective: To learn how to multiply two numbers when one is less than 10 and the other is greater than 10, but less than 100

Materials: 4 by 32 array for demonstration purposes

Vocabulary: Vertical form, partial products

Exploration:

> It is the purpose of this exploration to suggest ways of developing a computational scheme for multiplying. First have children associate a product expression with an appropriate array. This should provide the background helpful in learning a computational procedure.

We know how to find decimal numerals which name the same number, for example as 4×20 and 8×60. Now we want to learn how to find products of numbers like 4 and 32, 8 and 56, etc. One number will be less than 10. The other will be greater than 10 but less than 100.

Suppose we find the product of 4 and 32. We can think of 4×32 as the number of elements in an array of how many rows? (4) of how many columns? (32)

> Separate a 4 by 32 array into two arrays-- 4×30 and 4×2. Ask children to describe each. Your discussion about the arrays might develop as follows:

Now let us see if we can write what we've just done. We want to find a way of finding products without using an array each time.

To find n in $n = 4 \times 32$, we first renamed 32 as $30 + 2$. Why rename 32 as $30 + 2$ instead of, say $16 + 16$? (It is easier to find 4×30 and 4×2 than 4×16.) We can write the mathematical sentence as $n = 4 \times (30 + 2)$. Next, we used the distributive property of multiplication. ($n = (4 \times 30) + (4 \times 2)$) Then, we found the products (4×30) and (4×2). ($n = 120 + 8$) We added $120 + 8$ to find that $n = 128$.

What we have done may be summarized in this way:

$$\begin{aligned} 4 \times 32 &= 4 \times (30 + 2) && \text{(Rename } 32.) \\ &= (4 \times 30) + (4 \times 2) && \text{(Use the distributive property.)} \\ &= 120 + 8 && \Big(\text{Multiply } (4 \times 30) + (4 \times 2).\Big) \\ &= 128 && \text{(Add } 120 + 8.) \end{aligned}$$

We have been finding products using the multiplication facts and the distributive property. For each step we have been writing a new mathematical sentence. This way helps us to see why our answer is correct. Writing everything we think leaves us little to remember. It will be much quicker for us to find short cuts, but they will make us remember more. One way is to write our work like this. We will call this a vertical form.

$$\begin{array}{rl} 32 & \\ \underline{\times\ 4} & \\ 120 & \quad (4 \times 30) \\ \underline{\quad 8} & \quad (4 \times 2) \\ 128 & \end{array}$$

What do we think in writing this exercise?

Why do we write 120 and 8? (We think of $4 \times 32 = 4 \times (30 + 2) = (4 \times 30) + (4 \times 2)$. Why is this form a shorter way to write multiplication? (There are fewer symbols to write. It makes the addition easier.) How is it harder? (We have to think $32 = 30 + 2$ and remember it. We have to know $4 \times 30 = 120$ without writing any steps.)

We can write it in this way.

$$\begin{array}{r} 32 \\ \underline{\times\ 4} \end{array} \qquad \text{or} \qquad \begin{array}{r} 30 + 2 \\ \underline{\quad \times\ 4} \\ 120 + 8 = 128 \end{array}$$

Sometimes some children find this easier to understand.

The numbers 120 and 8 are called partial products in the multiplication. They must be added to get the final product, 128.

Let us try another example. $5 \times 61 = n$.
We can write

$$\begin{array}{r} 61 \\ \underline{\times 5} \end{array} \qquad \text{or} \qquad \begin{array}{r} 60 + 1 \\ \underline{\times 5} \\ 300 + 5 = 305 \end{array} \qquad \text{or} \qquad \begin{array}{rl} 61 & \\ \underline{\times 5} & \\ 300 & (5 \times 60) \\ \underline{5} & (5 \times 1) \\ 305 & \end{array}$$

This shows that $5 \times 61 = 300 + 5$. Why is this true? $5 \times 61 = 5 \times (60 + 1) = (5 \times 60) + (5 \times 1)$. We renamed 61 as $60 + 1$.

> These are some of the observations children should make. We rename a factor like 34 and 61 and use the distributive property. We regroup the factor into tens and ones, etc.; e.g., 32 was renamed $30 + 2$ and 61 was renamed $60 + 1$.
>
> It may be helpful to add an example in which a different renaming is shown, e.g.
>
> $$\begin{array}{r} 28 \\ \underline{\times 4} \end{array} \qquad \text{or} \qquad \begin{array}{r} 25 + 3 \\ \underline{\times 4} \\ 100 + 12 \end{array} \quad = \quad 112$$
>
> The class might be asked what mathematical sentences this abbreviates and why someone might write it (presumably because he happens to remember that $4 \times 25 = 100$). The main point should be the advantage of renaming 28 as $20 + 8$ rather than $25 + 3$. (Our numeral system shows the grouping into tens and ones, and it is easy to learn the few simple products 4×20, 4×30, ..., etc.)

Here is still another problem. 5×61. (Try several ways of finding the product.)

$$\begin{array}{r} 61 \\ \underline{\times 5} \\ \end{array} = \begin{array}{r} 60 + 1 \\ \underline{\times 5} \\ 300 + 5 = 305 \end{array} \quad \text{or} \quad \begin{array}{rl} 61 & \\ \underline{\times 5} & \\ 300 & (5 \times 60) \\ \underline{5} & (5 \times 1) \\ 305 & \end{array} \quad \text{or} \quad \begin{array}{rl} 61 & \text{(Note change} \\ \underline{\times 5} & \text{in order.)} \\ 5 & (5 \times 1) \\ \underline{300} & (5 \times 60) \\ 305 & \end{array}$$

For what mathematical sentences does this stand?

$$\begin{aligned} 5 \times 61 &= 5 \times (60 + 1) \\ &= (5 \times 60) + (5 \times 1) \\ &= 300 + 5 \\ &= 305 \end{aligned}$$

This shows that we can use either way and be correct.

Can anyone think of even a shorter way? Do we have to write the partial products before we write the product? What must we think if we leave them out? Look at $\begin{array}{r} 61 \\ \underline{\times 5} \end{array}$. First of all, I can think $5 \times 1 = 5$ and write the "5" because the other partial product 5×60 will end in "0". $\begin{array}{r} 61 \\ \underline{\times 5} \end{array}$ Instead of thinking $5 \times 60 = 300$, I can think $5 \times 60 = 30$ tens. 30 tens plus 5 ones is written 305, so I write

$$\begin{array}{r} 61 \\ \underline{\times 5} \\ 305 \end{array}$$

> Several additional illustrations of this thought process and its justification may be desirable here. However, don't push children to use this form at this level.

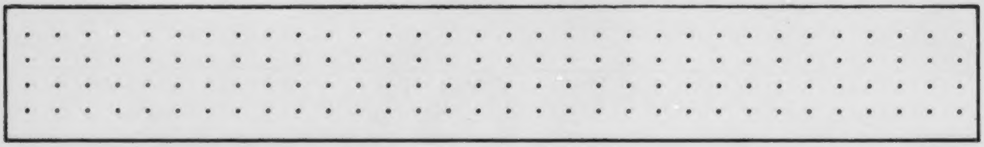
In Exercise Set 6 there are some exercises for practice in this short method.

MORE ABOUT MULTIPLYING

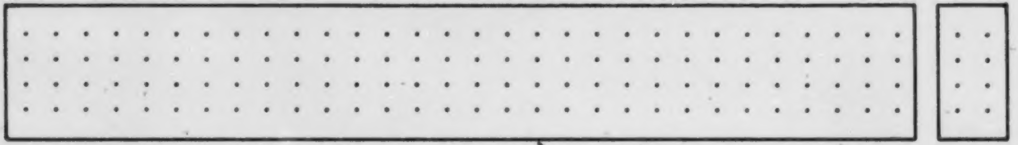
We now learn to multiply two numbers like these:

4 and 32

This array helps us to think about 4×32.



We can make smaller arrays.



How many rows does each have? (4) How many columns does each have? (30, 2) We write

$$\begin{aligned} 4 \times 32 &= 4 \times (30 + 2) \\ &= (4 \times 30) + (4 \times 2) \\ &= 120 + 8 \\ &= 128 \end{aligned}$$

We can use one of these ways too.

$$\begin{array}{r} 32 \\ \underline{\times\ 4} \end{array} \quad \text{or} \quad \begin{array}{r} 30 + 2 \\ \underline{\times\ 4} \\ 120 + 8 = 128 \end{array} \quad \text{or}$$

$$\begin{tikzcd}
32 & & 32 \\
\underline{\times\ 4} & & \underline{\times\ 4} \\
120 & (4 \times 30) \arrow[l] \arrow[dr] & 8 \\
\underline{8} & (4 \times 2) \arrow[l] \arrow[ur] & \underline{120} \\
128 & & 128
\end{tikzcd}$$

Can you think of another way?

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Exercise Set 6

Find the decimal numeral for each product in the following exercises. Use two forms as in the example.

Example:

$$\begin{aligned} 3 \times 12 &= 3 \times (10 + 2) \\ &= (3 \times 10) + (3 \times 2) \\ &= 30 + 6 \\ &= 36 \end{aligned} \qquad \begin{array}{r} 12 \\ \times 3 \\ \hline 6 \\ 30 \\ \hline 36 \end{array}$$

1. 3×32

$$\begin{aligned} 3 \times 32 &= 3 \times (30+2) \\ &= (3 \times 30) + (3 \times 2) \\ &= 90 + 6 \\ &= 96 \end{aligned} \qquad \begin{array}{r} 32 \\ \times 3 \\ \hline 6 \\ 90 \\ \hline 96 \end{array}$$

2. 4×23

$$\begin{aligned} 4 \times 23 &= 4 \times (20+3) \\ &= (4 \times 20) + (4 \times 3) \\ &= 80 + 12 \\ &= 92 \end{aligned} \qquad \begin{array}{r} 23 \\ \times 4 \\ \hline 12 \\ 80 \\ \hline 92 \end{array}$$

3. 6×34

$$\begin{aligned} 6 \times 34 &= 6 \times (30+4) \\ &= (6 \times 30) + (6 \times 4) \\ &= 180 + 24 \\ &= 204 \end{aligned} \qquad \begin{array}{r} 34 \\ \times 6 \\ \hline 24 \\ 180 \\ \hline 204 \end{array}$$

4. 4×65

$$\begin{aligned} 4 \times 65 &= 4 \times (60+5) \\ &= (4 \times 60) + (4 \times 5) \\ &= 240 + 20 \\ &= 260 \end{aligned} \qquad \begin{array}{r} 65 \\ \times 4 \\ \hline 20 \\ 240 \\ \hline 260 \end{array}$$

5. 4×82

$$\begin{aligned} 4 \times 82 &= 4 \times (80+2) \\ &= (4 \times 80) + (4 \times 2) \\ &= 320 + 8 \\ &= 328 \end{aligned} \qquad \begin{array}{r} 82 \\ \times 4 \\ \hline 8 \\ 320 \\ \hline 328 \end{array}$$

Exercise Set 6 (Cont'd)

6. $5 \times 87 = 5 \times (80+7)$
$= (5 \times 80) + (5 \times 7)$
$= 400 + 35$
$= 435$

$$\begin{array}{r} 87 \\ \times\ 5 \\ \hline 35 \\ 400 \\ \hline 435 \end{array}$$

7. $7 \times 34 = 7 \times (30+4)$
$= (7 \times 30) + (7 \times 4)$
$= 210 + 28$
$= 238$

$$\begin{array}{r} 34 \\ \times\ 7 \\ \hline 28 \\ 210 \\ \hline 238 \end{array}$$

8. $8 \times 37 = 8 \times (30+7)$
$= (8 \times 30) + (8 \times 7)$
$= 240 + 56$
$= 296$

$$\begin{array}{r} 37 \\ \times\ 8 \\ \hline 56 \\ 240 \\ \hline 296 \end{array}$$

9. $4 \times 36 = 4 \times (30+6)$
$= (4 \times 30) + (4 \times 6)$
$= 120 + 24$
$= 144$

$$\begin{array}{r} 36 \\ \times\ 4 \\ \hline 24 \\ 120 \\ \hline 144 \end{array}$$

10. $8 \times 89 = 8 \times (80+9)$
$= (8 \times 80) + (8 \times 9)$
$= 640 + 72$
$= 712$

$$\begin{array}{r} 89 \\ \times\ 8 \\ \hline 72 \\ 640 \\ \hline 712 \end{array}$$

MULTIPLYING LARGER NUMBERS

Objective: To extend the ideas used to multiply a number with a "two-digit" numeral by a number with a "one-digit" numeral to ways which may be used to multiply numbers with "three-digit" and "four-digit" numerals by a "one-digit" numeral

First, you may wish to review how children have learned to multiply two numbers such as 3 and 74, that is,

$$\begin{aligned} 3 \times 74 &= 3 \times (70 + 4) \\ &= (3 \times 70) + (3 \times 4) \\ &= 210 + 12 \\ &= 222 \end{aligned}$$

or

$$\begin{array}{rl} 74 & \\ \underline{\times\ 3} & \\ 210 & (3 \times 70) \\ \underline{12} & \underline{(3 \times 4)} \\ 222 & (3 \times 74) \end{array}$$

Then let them see how they could use the same procedures for numbers like 3 and 312. Have them suggest how they might rename 312. Then ask how they might use the distributive property.

Note that a good way is $300 + 10 + 2$ because we know products like 3×100 and 3×10.

$$\begin{aligned} 3 \times 312 &= 3 \times (300 + 10 + 2) \\ &= (3 \times 300) + (3 \times 10) + (3 \times 2) \\ &= 900 + 30 + 6 \\ &= 936 \end{aligned}$$

Then explore possible ways using the vertical form.

Vertical Forms:

$$\begin{array}{r} 300 + 10 + 2 \\ \times 3 \\ \hline 900 + 30 + 6 = 936 \end{array}$$

or

$$\begin{array}{rl} 312 & \\ \times 3 & \\ \cline{1-1} 900 & (3 \times 300) \\ 30 & (3 \times 10) \\ 6 & (3 \times 2) \\ \hline 936 & (3 \times 312) \end{array}$$

or

$$\begin{array}{r} 312 \\ \times 3 \\ \hline 6 \\ 30 \\ 900 \\ \hline 936 \end{array}$$

Some children may find the short form

$$\begin{array}{r} 312 \\ \times 3 \\ \hline 936 \end{array}$$

easy for them. At this time, it is probably not the best form for all children.

Use other pairs of numbers, including among them pairs like 301 and 2, 6 and 1211, etc.

$$\begin{aligned} 2 \times 301 &= 2 \times (300 + 1) \\ &= (2 \times 300) + (2 \times 1) \\ &= 600 + 2 \\ &= 602 \end{aligned}$$

Vertical Forms:

$$\begin{array}{rl} 301 & \\ \times 2 & \\ \cline{1-1} 600 & (2 \times 300) \\ 2 & (2 \times 1) \\ \hline 602 & (2 \times 301) \end{array}$$

or

$$\begin{array}{r} 301 \\ 2 \\ \hline 2 \\ 600 \\ \hline 602 \end{array}$$

$$\begin{aligned} 6 \times 1211 &= 6 \times (1000 + 200 + 10 + 1) \\ &= (6 \times 1000) + (6 \times 200) + (6 \times 10) + \\ &\quad (6 \times 1) \\ &= 6000 + 1200 + 60 + 6 \\ &= 7266 \end{aligned}$$

Vertical Forms:

$$\begin{array}{rl} 1211 & \\ \times 6 & \\ \cline{1-1} 6000 & (6 \times 1000) \\ 1200 & (6 \times 200) \\ 60 & (6 \times 10) \\ 6 & (6 \times 1) \\ \hline 7266 & (6 \times 1211) \end{array}$$

or

$$\begin{array}{r} 1211 \\ 6 \\ \hline 6 \\ 60 \\ 1200 \\ 6000 \\ \hline 7266 \end{array}$$

MULTIPLYING LARGER NUMBERS

We know how to find the products of numbers like 3 and 46, 7 and 39, 6 and 45. What number must n represent in each of these sentences if the sentence is a true statement?

$$3 \times 46 = n$$

$$7 \times 39 = n$$

$$6 \times 45 = n$$

The products are 273, 138, and 270. Now match the products and the product expressions.

We now want to find the product of numbers like 3 and 312.

We write

$$\begin{aligned} 3 \times 312 &= 3 \times (300 + 10 + 2) \\ &= (3 \times 300) + (3 \times 10) + (3 \times 2) \\ &= 900 + 30 + 6 \\ &= 936 \end{aligned}$$

There are several ways that we might use the vertical form for multiplication. Here are some of them.

$$\begin{array}{r} 300 + 10 + 2 \\ \times 3 \\ \hline 900 + 30 + 6 = 936 \end{array}$$

or

$$\begin{array}{rl} 312 & \\ \underline{\times 3} & \\ 900 & \longleftarrow (3 \times 300) \\ 30 & \longleftarrow (3 \times 10) \\ \underline{6} & \longleftarrow \underline{(3 \times 2)} \\ 936 & \longleftarrow (3 \times 312) \end{array}$$

or

$$\begin{array}{r} 312 \\ \underline{\times 3} \\ 6 \\ 30 \\ \underline{900} \\ 936 \end{array}$$

In the last example, how did we get 900, 30, and 6?

You do not need to use all of these ways. Use the one that you like best. You may even like a short form like this.

$$\begin{array}{r} 312 \\ \underline{\times 3} \\ 936 \end{array}$$

Can you discover a way to find the product of 4 and 2102?

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Exercise Set 7

Find the decimal numeral for each product in the following exercises. Show the partial products.

1. 2 × 311
(622)

2. 2 × 434
(868)

3. 4 × 322
(1288)

4. 3 × 412
(1236)

5. 3 × 210
(630)

6. 2 × 303
(606)

7. 4 × 300
(1200)

8. 5 × 601
(3005)

9. 8 × 711
(5688)

10. 3 × 3020
(9060)

11. 4 × 3002
(12,008)

12. 7 × 5101
(35,707)

Exercise Set 8

Work these exercises as in the example. Use the vertical form if you can.

Example:

311
× 5
1,555

1. 243
× 2
486

2. 210
× 4
840

3. 203
× 3
609

4. 202
× 4
808

5. 420
× 2
840

6. 800
× 6
4800

7. 821
× 4
3284

8. 3020
× 3
9060

9. 3002
× 4
12,008

10. 502
× 3
1506

11. 134
× 2
268

12. 612
× 4
2448

13. 723
× 3
2169

14. 632
× 3
1896

15. 734
× 2
1468

16. 1010
× 9
9090

17. 1023
× 3
3069

18. 2332
× 3
6996

19. 8212
× 4
32,848

20. 9111
× 8
72,888

A SHORTER METHOD OF MULTIPLYING

Objective: To help children (who are ready for it) develop a shorter way of multiplying

> You may wish to delay this lesson. Yet, it seems desirable to make a conscious effort to express the product of two numbers such as 4 and 34 as 136 instead of first expressing it as the sum of two products 120 and 16, and then expressing this sum as 136. However, there is nothing wrong with doing the latter if children are not ready to try the shorter method.

Again, let us look at a problem like: $4 \times 34 = n$. How can we find the number n represents?

> If children do not suggest the shorter form, then suggest this as another possibility. You might find these ideas helpful in the development. We can write the steps in multiplying 34 by 4 in this way:
>
> $$\begin{array}{r l} 34 & \\ \underline{\times\ 4} & \\ 16 & (4 \times 4) \\ \underline{120} & \underline{(4 \times 30)} \\ 136 & (4 \times 34) \end{array}$$

Suppose we see if we can find a shorter method.

$$\begin{array}{r} 34 \\ \underline{\times\ 4} \end{array}$$

$4 \times 4 = 16$. Think of this as 1 ten and 6 ones. Instead of writing 16, write 6 in the one's place. We will remember the 1 ten or we can write 1 above the 3 (indicate this).

$$\begin{array}{r} \overset{1}{3}4 \\ \underline{\times\ 4} \\ 6 \end{array}$$

You also said $4 \times 30 = 120$. This is how many tens? (12) But what haven't we included yet? (The one ten from the 16). So 12 tens and 1 ten is 13 tens or 130. How do you write the numeral for 13 tens and 6 ones? (136) What is the product of 4 and 34? (136)

> Try several other examples such as 5×37, 3×26, 6×45, etc. When children are not ready, do not insist upon this short cut. It may be that they will not be ready to use it until later. Even if they do seem ready for some examples, use both the long and short forms side by side to show clearly what is being remembered.

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A SHORTER METHOD OF MULTIPLYING

In our last lesson you saw how to shorten the form at the left so that you could multiply 312 by 3 in the way shown at the right:

$$\begin{array}{r} 312 \\ \underline{\times\ 3} \\ 6 \\ 30 \\ \underline{900} \\ 936 \end{array} \qquad\qquad \begin{array}{r} 312 \\ \underline{\times\ 3} \\ 936 \end{array}$$

Were you able to tell what you had to think to use this way? We want to think some more about these shorter ways.

Suppose we need to find the product of 4 and 23. We already know how to find the product in this way:

$$\begin{array}{r} 23 \\ \underline{\times\ 4} \\ 12 \\ \underline{80} \\ 92 \end{array}$$

Now let us see if we can find a shorter form for doing this.

$$\begin{array}{r} 23 \\ \underline{\times\ 4} \\ 2 \end{array}$$

We can think $4 \times 3 = 12$. Think of 12 as 1 ten and 2 ones. Let us write the 2 in the one's place. We will remember the 1 ten.

$$\begin{array}{r} 23 \\ \underline{\times\ 4} \\ 92 \end{array}$$

$4 \times 20 = 80$. 80 is 8 tens. 8 tens and 1 ten are 9 tens. How can we write the numeral for 9 tens and 2 ones?

What is the product of 4 and 23? We can write

$$4 \times 23 = 92.$$

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Here are some problems. Can you fill in the blanks?

$$\begin{array}{r} 45 \\ \underline{\times\ 3} \\ 13\underline{5} \end{array} \qquad\qquad \begin{array}{r} 82 \\ \underline{\times\ 6} \\ 4\underline{9}2 \end{array}$$

$$\begin{array}{r} 37 \\ \underline{\times\ 5} \\ \underline{18}5 \end{array} \qquad\qquad \begin{array}{r} 74 \\ \underline{\times\ 9} \\ 66\underline{6} \end{array}$$

$$\begin{array}{r} 28 \\ \underline{\times\ 3} \\ \underline{8}4 \end{array}$$

You may wish to use this short method to do the problems in Exercise Set 9. You may wish to try several methods.

Exercise Set 9

Use the way you like best to find the number represented by n in each of these.

1. 4 × 17 = n (n= 68)
2. 7 × 23 = n (n= 161)
3. 9 × 62 = n (n= 558)
4. 6 × 81 = n (n= 486)
5. 7 × 87 = n (n= 609)
6. 9 × 56 = n (n= 504)
7. 5 × 52 = n (n= 260)
8. 3 × 89 = n (n= 267)
9. 6 × 56 = n (n= 336)
10. 8 × 78 = n (n= 624)

Copy and complete. (Note: Each blank must be replaced by one digit.)

11. 45
× 5
225

12. 38
× 6
228

13. 59
× 7
413

14. 79
× 5
395

15. 18
× 4
72

Note: The product is a two-digit numeral.

16. 45
× 8
360

17. 1__
× 6
6__

(100, 101, 102, ..., 116. The product varies as the missing factor varies.)

MULTIPLYING NUMBERS LESS THAN 100 BY MULTIPLES OF 10

Objective: To develop an understanding of how to find the product of numbers like 23 and 30, 50 and 46, etc.; that is, where one number is a multiple of 10 and the other is greater than 10 but less than 100

Recall with the children how they found the product of 23 and 3, that is,

$$\begin{array}{rl} 23 & \\ \times\ 3 & \\ \cline{1-1} 69 & (3 \times 3) + (3 \times 20) \end{array}$$

Some pupils still may be working at this level.

$$\begin{array}{rl} 23 & \\ \times\ 3 & \\ \cline{1-1} 60 & (3 \times 20) \\ 9 & (3 \times 3) \\ \hline 69 & (3 \times 23) \end{array} \qquad \underline{\underline{\text{or}}} \qquad \begin{array}{r} 23 \\ \times\ 3 \\ \hline 9 \\ 60 \\ \hline 69 \end{array}$$

Some still may prefer to use this method.

$$\begin{aligned} 3 \times 23 &= 3 \times (20 + 3) \\ &= (3 \times 20) + (3 \times 3) \\ &= 60 + 9 \\ &= 69 \end{aligned}$$

Ask children to suggest possible procedures for finding 30×23 from what they already know. For example, some may suggest the idea of renaming one of the factors.

$$\begin{aligned} 30 \times 23 &= 30 \times (20 + 3) \\ &= (30 \times 20) + (30 \times 3) \end{aligned}$$

We know how to find these products.

$$30 \times 20 = 600 \quad \text{and} \quad 30 \times 3 = 90.$$

You may need to provide extra practice in finding products of two multiples of 10 like 20 and 30, 40 and 30, etc. Or, some may suggest the use of the vertical form. The discussion might go something like this.

$$30 \times 23 = n$$

Since we can express 23 as $20 + 3$,

$$\begin{aligned} 30 \times 23 &= 30 \times (20 + 3) \\ &= (30 \times 20) + (30 \times 3) \\ &= 600 + 90 \\ &= 690 \end{aligned} \qquad \begin{array}{rl} 23 & \\ \times 30 & \\ \cline{1-1} 90 & (30 \times 3) \\ 600 & (30 \times 20) \\ \hline 690 & (30 \times 23). \end{array}$$

Still another approach is to recall how we can multiply any number by 10. For example, 23×10 means 23 tens for which we write the numeral 230. So, $23 \times 10 = 230$.

We can think of 30×23 as $(10 \times 3) \times 23$. We can first find the product of 3 and 23. Then we can multiply that product by 10.

$$\begin{aligned} 30 \times 23 &= (10 \times 3) \times 23 \\ &= 10 \times (3 \times 23) \\ &= 10 \times 69 \\ &= 690 \end{aligned}$$

All children may not be ready to use a shorter form

$$\begin{array}{r} 23 \\ \underline{\times\ 30} \\ 690 \end{array}$$

In the background of their learning experiences an exploration such as this will help them to develop a shorter procedure at a later date.

Think through examples such as the following with your class, using those procedures from above which seem most appropriate for your pupils.

$$\begin{array}{r} 34 \\ \underline{\times\ 20} \end{array} \qquad \begin{array}{r} 32 \\ \underline{\times\ 40} \end{array} \qquad \begin{array}{r} 18 \\ \underline{\times\ 50} \end{array} \qquad \begin{array}{r} 63 \\ \underline{\times\ 70} \end{array}$$

We suggest, for example:

Expressing 34 as $(30 + 4)$. Then

$$\begin{array}{l} 30 \quad + \quad 4 \\ \underline{\qquad\qquad \times\ 20} \\ 600 \quad + 80 = 680 \end{array} \qquad \underline{\text{or}} \qquad \begin{array}{rl} 34 & \\ \underline{\times\ 20} & \\ 80 & (20 \times 4) \\ \underline{600} & \underline{(20 \times 30)} \\ 680 & (20 \times 34) \end{array}$$

Encourage children who are ready to find the product using this form.

$$\begin{array}{r} 34 \\ \underline{\times\ 20} \\ 80 \\ \underline{600} \\ 680 \end{array}$$

Some children may want to use even the shorter form.

$$\begin{array}{r} 34 \\ \underline{\times\ 20} \\ 680 \end{array}$$

This might be fostered by thinking first 2×34 and then multiplying that product by 10.

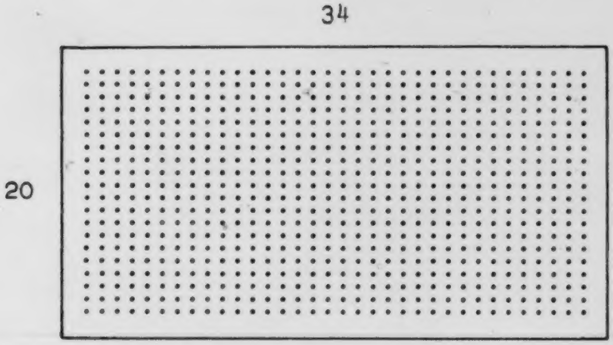
MULTIPLYING NUMBERS LESS THAN 100 BY MULTIPLES OF 10

We now want to learn how to multiply numbers like 20 and 34, 40 and 46, 50 and 23, 60 and 31.

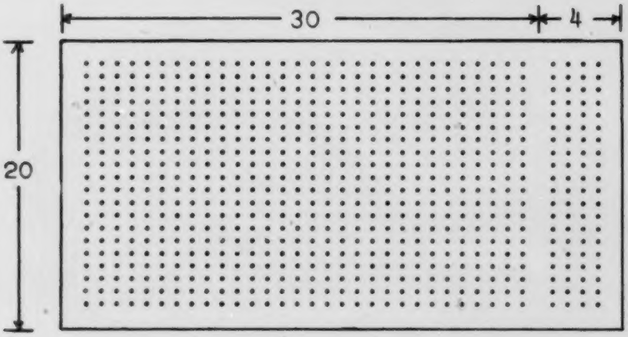
Is one number of each pair a multiple of 10? Name the multiples of 10 in these pairs.

Is the other number in each pair less than 100?

Let us learn a way to find the product of 20 and 34. Does 20 × 34 name the number of dots in this array? How can you tell?



Here is another array just like the one above. Let us separate it into smaller arrays.



Does (20 × 34) describe the first array? (yes)

Does (20 × 30) + (20 × 4) describe the second array? (yes)

Do the two arrays show that 20 × 34 = (20 × 30) + (20 × 4)? (yes)

Does this help you to find the number of dots in the big array? (yes)

We can write:

$$\begin{aligned} 20 \times 34 &= 20 \times (30 + 4) \\ &= (20 \times 30) + (20 \times 4) \end{aligned}$$

$$\begin{array}{rcl} 30 & + & 4 \\ & & \times\ 20 \\ \hline 600 & + & 80 = 680 \end{array}$$

$$\begin{array}{rl} 34 & \\ \underline{\times\ 20} & \\ 80 & (20 \times 4) \\ \underline{600} & \underline{(20 \times 30)} \\ 680 & (20 \times 34) \end{array}$$

Can you think of other ways of finding the product of these two numbers?

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Exercise Set 10

Find the number n represents for each of these.

1. $30 \times 30 = n$
(n= 900)

2. $20 \times 40 = n$
(n=800)

3. $10 \times 80 = n$
(n=800)

4. $30 \times 40 = n$
(n=1200)

5. $20 \times 50 = n$
(n=1000)

6. $60 \times 90 = n$
(n=5400)

7. $70 \times 70 = n$
(n=4900)

8. $80 \times 40 = n$
(n= 3200)

9. $30 \times 90 = n$
(n= 2700)

10. $50 \times 70 = n$
(n=3500)

11. $20 \times 60 = n$
(n=1200)

12. $70 \times 80 = n$
(n= 5600)

13. $20 \times 43 = n$
(n= 860)

14. $30 \times 33 = n$
(n= 990)

15. $10 \times 87 = n$
(n=870)

16. $50 \times 32 = n$
(n= 1600)

17. $50 \times 62 = n$
(n= 3100)

18. $70 \times 83 = n$
(n= 5810)

19. $90 \times 65 = n$
(n= 5850)

20. $80 \times 87 = n$
(n= 6960)

21. $90 \times 38 = n$
(n= 3420)

22. $70 \times 57 = n$
(n=3990)

23. $40 \times 93 = n$
(n=3720)

24. $60 \times 83 = n$
(n=4980)

FINDING PRODUCTS OF ANY TWO NUMBERS GREATER THAN 10 (AND LESS THAN 100)

Objective: To learn ways of finding products of pairs of numbers like 23 and 34, 46 and 27, etc., where both numbers are less than 100, more than 10, but neither a multiple of 10

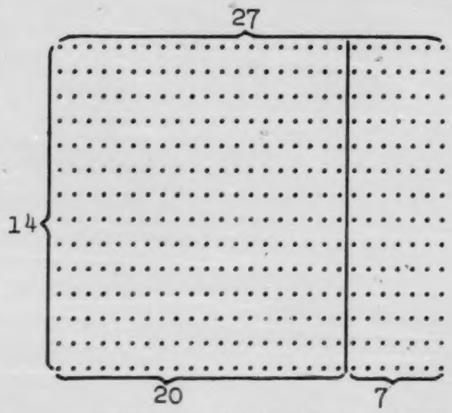
Materials: A 14 by 27 array made of sturdy material which can be folded

Other arrays in which both the number of rows and columns are represented by two-digit numerals

You may use arrays to good advantage to aid pupils in understanding the necessary steps in multiplying, for example, 14 by 27. You might use either, or both, of the following methods. Observe that each method depends upon the distributive property. In the first method, (both forms A and B) the distributive property is used once; in the second method, it is used three times.

We want to multiply 14 by 27.

FIRST METHOD



14 by 27 array:

14 × 20 = 280
14 × 7 = 98
280 + 98 = 378

There are 378 dots in the total array.

Observe with the pupils that the number of dots in the array is obtained by multiplying 14 by 27. (There are 14 rows and 27 columns.) Hence 14 × 27 = number of dots.

Then separate the array as indicated. The number of dots in one part is 14 × 20. The number in the other part is 14 × 7. Hence,

(A)

$$\begin{aligned} 14 \times 27 &= 14 \times (20 + 7) \\ &= (14 \times 20) + (14 \times 7 \\ &= 280 + 98 \\ &= 378 \end{aligned}$$

We would hope that many of the pupils can multiply 14 by 20 and 14 by 7 from what they have learned in preceding parts of this chapter. It is important that in finding the partial products, the pupils recognize they are using many of the multiplication techniques they have learned.

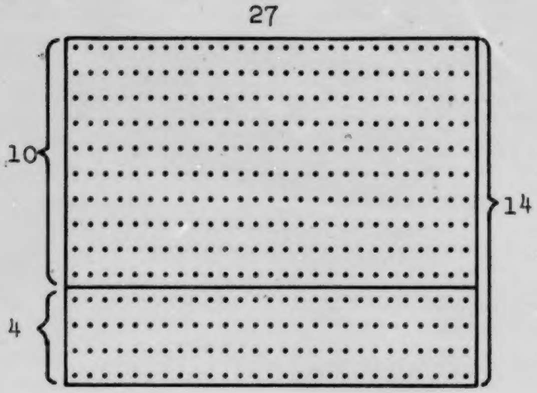
After the array has been folded, the work should be recorded on the chalkboard. The pupils should be introduced to both of the following methods of writing the partial products.

Form (A):

$$\begin{aligned} 14 \times 27 &= 14 \times (20 + 7) \\ &= (14 \times 20) + (14 \times 7) \\ &= 280 + 98 \\ &= 378 \end{aligned}$$

$$\begin{array}{rl} 27 & \\ \times\ 14 & \\ \cline{1-1} 280 & (14 \times 20) \\ 98 & \underline{(14 \times 7)} \\ \cline{1-1} 378 & (14 \times 27) \end{array}$$

Consider this array for the same example.



$$\begin{aligned}
10 \times 27 &= 270 \\
4 \times 27 &= 108 \\
14 \times 27 &= (10 + 4) \times 27 \\
&= (10 \times 27) + (4 \times 27) \\
&= 270 + 108 \\
&= 378
\end{aligned}$$

Again after the array has been folded, work should be recorded on the chalkboard.

$$\begin{aligned}
14 \times 27 &= (10 + 4) \times 27 \\
&= (10 \times 27) + (4 \times 27) \\
&= 270 + 108 \\
&= 378
\end{aligned}$$

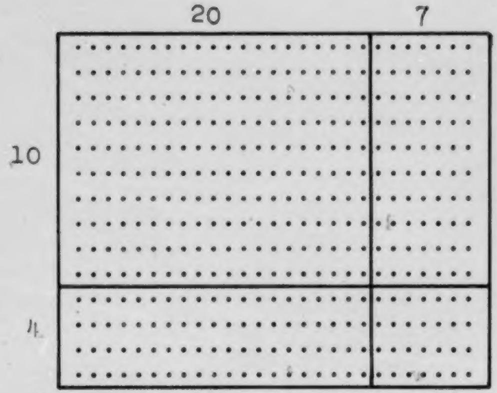
Or changing order

$$\begin{array}{rl}
27 & \\
\underline{\times\ 14} & \\
270 & (10 \times 27) \\
\underline{108} & \underline{(4 \times 27)} \\
378 & (14 \times 27)
\end{array}
\qquad\qquad
\begin{array}{rl}
27 & \\
\underline{\times\ 14} & \\
108 & (4 \times 27) \\
\underline{270} & \underline{(10 \times 27)} \\
378 & (14 \times 27)
\end{array}$$

There is an advantage to be gained by consideration of a second method even if all the pupils should be able to understand readily the first method.

SECOND METHOD

$$\begin{aligned} 14 \times 27 &= 14 \times (20 + 7) \\ &= (14 \times 20) + (14 \times 7) \end{aligned}$$



The pupils should be helped (if necessary) to understand that

$$14 \times 20 = (10 + 4) \times 20 \quad \text{and}$$

$$14 \times \ 7 = (10 + 4) \times 7.$$

Then write

$$\begin{aligned} 14 \times 27 &= 14 \times (20 + 7) \\ &= (14 \times 20) + (14 \times 7) \\ &= [(10 + 4) \times 20] + [(10 + 4) \times 7] \\ &= [(10 \times 20) + (4 \times 20)] + \\ &\quad\ [(10 \times 7) + (4 \times 7)] \\ &= [200 + 80] + [70 + 28] \\ &= 280 + 98 \\ &= 378 \end{aligned}$$

The partial products 200, 80, 70, and 28 should now be related to the number of dots in the 4 partitions of the array.

$$\begin{array}{r l} 27 & \\ \times\ 14 & \\ \hline 28 & (4 \times 7) \\ 80 & (4 \times 20) \\ 70 & (10 \times 7) \\ 200 & (10 \times 20) \\ \hline 378 & (14 \times 27) \end{array}$$

Several similar examples with which appropriate arrays and both methods are used may be necessary.

FINDING PRODUCTS OF NUMBERS GREATER THAN 10 (AND LESS THAN 100)

We have learned to find the product of pairs of numbers. We made some choices in the numbers we selected.

We learned to find products of pairs of numbers like these:

3 and 45

$$\begin{aligned} 3 \times 45 &= 3 \times (40 + 5) \\ &= (3 \times 40) + (3 \times 5) \\ &= 120 + 15 \\ &= 135 \end{aligned}$$

8 and 16

$$\begin{aligned} 8 \times 16 &= 8 \times (10 + 6) \\ &= (8 \times 10) + (8 \times 6) \\ &= 80 + 48 \\ &= 128 \end{aligned}$$

$$\begin{array}{rl} 45 & \\ \underline{\times\ 3} & \\ 120 & (3 \times 40) \\ \underline{15} & (3 \times 5) \\ 135 & \end{array}$$

$$\begin{array}{rl} 16 & \\ \underline{\times\ 8} & \\ 128 & (8 \times 6) + (8 \times 10) \end{array}$$

5 and 24

$$\begin{aligned} 5 \times 24 &= 5 \times (20 + 4) \\ &= (5 \times 20) + (5 \times 4) \\ &= 100 + 20 \\ &= 120 \end{aligned}$$

$$\begin{array}{rl} 24 & \\ \underline{\times\ 5} & \\ 120 & (5 \times 4) + (5 \times 20) \end{array}$$

We can use the same way for all of these examples. We can use different ways to find the product of numbers like these.

We learned to find products of numbers when one of the numbers was a multiple of 10. What numbers less than 100 are multiples of 10?

Can you find the product for each of these pairs?

20 and 45 10 and 17

$$\begin{array}{r} 45 \\ \underline{\times\ 20} \end{array} \qquad \begin{array}{r} 17 \\ \underline{\times\ 10} \end{array}$$

37 and 20

$$\begin{array}{r} 20 \\ \underline{\times\ 37} \end{array} \quad \text{or} \quad \begin{array}{r} 17 \\ \underline{\times\ 20} \end{array}$$

Why can we change order?

The products are 900, 170, and 740. Did you get them right?

Now we will learn how to find the products of any two numbers greater than 10. We will still make a choice. They will be numbers less than 100.

Ann was cutting a large cake.

She cut 12 long pieces (rows) of cake.

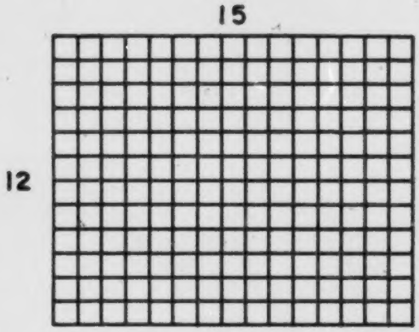
She cut each long piece into 15 pieces.

How many pieces of cake did she have?

Is this mathematical sentence a correct one for this problem?

$12 \times 15 = n$ Why?

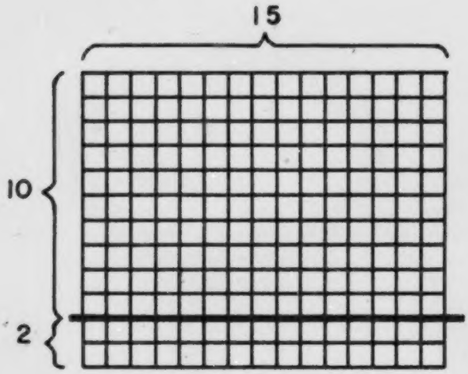
Does this picture give us an idea of the cake? Explain.



Let us separate this array into smaller arrays.

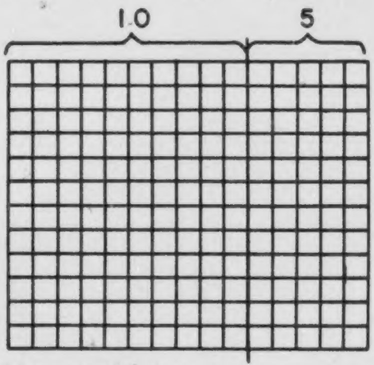
What product expression does each array suggest to you?

Find the array each of these describes.



$$\begin{array}{r}10 \times 15 = 150 \\ \underline{2 \times 15 = 30} \\ 12 \times 15 = 180\end{array}$$

There are other ways we can separate the array. Look at this one.



$12 \times 15 = 12 \times (10 + 5)$

$= (12 \times 10) + (12 \times 5)$

$= 120 + 60$

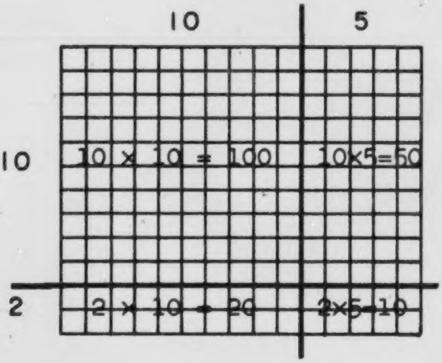
$= 180$

This vertical form helps us to find the product.

15	
× 12	
120	(12 × 10)
60	(12 × 5)
180	(12 × 15)

Let us see still another way of separating the array. Now we can write the product in vertical form in this way.

15	
× 12	
10	(2 × 5)
20	(2 × 10)
50	(10 × 5)
100	(10 × 10)
180	(12 × 15)



Use the way that you like best to find the products in the exercises in Exercise Set 11. (The more we can learn to remember, the less we need to write.)

Exercise Set 11

Find the decimal numeral representing n.

1. $12 \times 23 = n$
   ($n = 276$)

2. $11 \times 42 = n$
   ($n = 462$)

3. $14 \times 52 = n$
   ($n = 728$)

4. $15 \times 32 = n$
   ($n = 480$)

5. $32 \times 41 = n$
   ($n = 1{,}312$)

6. $23 \times 63 = n$
   ($n = 1{,}449$)

7. $21 \times 78 = n$
   ($n = 1{,}638$)

8. $31 \times 66 = n$
   ($n = 2{,}046$)

9. $32 \times 59 = n$
   ($n = 1{,}888$)

10. $45 \times 45 = n$
    ($n = 2{,}025$)

11. $37 \times 48 = n$
    ($n = 1{,}776$)

12. $29 \times 54 = n$
    ($n = 1{,}566$)

(Optional)

Sometimes properties of multiplication can be used to make short cuts in multiplication. See if you can explain these short cuts.

13. 20 × 78 short cut: rewrite or rethink as 78 × 20 = 1560

$$\begin{array}{r} 20 \\ \underline{\times\ 78} \end{array} \qquad \begin{array}{r} 78 \\ \underline{\times\ 20} \\ 1560 \end{array}$$

14. 33 × 29 short cut: write or think as

$(30 \times 33) - 33 = 957$

$$\begin{array}{r} 33 \\ \underline{\times\ 29} \end{array} \qquad \begin{array}{r} 990 \\ \underline{-\ 33} \\ 957 \end{array}$$

15. 101 × 78 short cut: use short cut or think as

$(100 \times 78) + 78 = 7878$

$$\begin{array}{r} 101 \\ \underline{\times\ 78} \end{array} \qquad \begin{array}{r} 7800 \\ \underline{+\ 78} \\ 7878 \end{array}$$

16. 480 × 370 short cut: find 37 × 48, then multiply by 100.

$$\begin{array}{r} 480 \\ \underline{\times\ 370} \end{array}$$

$(37 \times 48) \times 100 = 1776 \times 100$

$= 177{,}600$

Try to find a short cut in these exercises. If you can't, do them in the usual way.

17. 50 × 49

$$\begin{array}{r} 50 \\ \underline{\times\ 49} \end{array} \qquad \begin{array}{r} 49 \\ \underline{\times 50} \\ 2450 \end{array} \text{ or } \begin{array}{r} 2500 \\ \underline{-50} \\ 2450 \end{array}$$

19. 203 × 32

$$\begin{array}{r} 203 \\ \underline{\times\ 32} \end{array}$$

$(200 \times 32) + (3 \times 32)$

$= 6400 + 96$

$= 6496$

18. 30 × 86

$$\begin{array}{r} 30 \\ \underline{\times\ 86} \end{array} \qquad \begin{array}{r} 86 \\ \underline{\times 30} \\ 2580 \end{array}$$

20. 500 × 680

$$\begin{array}{r} 500 \\ \underline{\times\ 680} \end{array} \qquad \begin{array}{r} 680 \\ \underline{\times 500} \\ 340{,}000 \end{array}$$

USING MULTIPLICATION IN PROBLEM SOLVING

Objective: To help children further their ability to solve problems through the use of mathematical sentences requiring the multiplication of numbers named by more than one digit

The method of problem solving emphasized in Chapter 3 should be expanded and reinforced by the development here.

To summarize, emphasis is on the relationship in a problem. To solve problems, first identify the question that is to be answered. Then use the information given in the problem to aid in writing a mathematical sentence which expresses the relation between the information and the question to be answered. When the result is determined, use it to write an answer sentence.

USING MULTIPLICATION IN PROBLEM SOLVING

You have solved problems before.

Do you remember how you solved problems?

Let us use this problem to help us remember.

Problem: At the circus, the children of Madison School sat in a section of 15 rows. Eighteen children were seated in each row. How many children from Madison School were seated in this section?

Bits of Information: There are 15 rows and there are 18 children in each row.

Mathematical Sentence: $15 \times 18 = n$

Work:

$$\begin{array}{r} 18 \\ \underline{\times\ 15} \\ 40 \\ 50 \\ 80 \\ \underline{100} \\ 270 \end{array}$$

Answer Sentence: 270 children were seated in this section.

In solving problems you need to:

Understand the question that is to be answered.

Find the information given in the problem that will help you.

Write a mathematical sentence that relates this information to the question.

Find the number that is not known.

Write an answer to the problem question.

Exercise Set 12

1. The children of Madison School went to the circus in 6 buses. **Forty-five** children rode in each bus. How many children rode in the 6 buses? $s = 6 \times 45$ $= 270$
270 children rode in 6 buses.

2. There were 424 boys and girls enrolled in Madison School. If 270 children went to the circus and the other children went to the zoo, how many children went to the zoo?
$n = 424 - 270$ $= 154$ 154 children went to the zoo.

3. One day at the zoo there were 154 children from Madison School and 168 children from Adams School. How many children visited the zoo that day from the two schools?
$n = 154 + 168$ $= 322$ 322 children visited the zoo.

4. A crossword puzzle had 15 squares across and 12 squares down. How many squares were there in the puzzle?
$n = 15 \times 12$ $= 180$ The crossword puzzle contained 180 squares.

5. There were 360 dots in one part of an array and 24 dots in the other part. How many dots were there in the whole array? $s = 360 + 24$ $= 384$ There were 384 dots in the array.

6. The score of a football game is 35 to 17. How many points does one team need to tie the score? $t = 35 - 17$ $= 18$
One team needs 18 points to tie the score.

7. Mrs. Smith buys 14 gallons of milk each month. How many gallons does she buy in a year? $n = 14 \times 12$ $= 168$
Mrs. Smith buys 168 gallons of milk in a year.

8. Fifteen gallons of ice cream were bought for the Halloween party. If one gallon served 26 children, how many children did the 15 gallons serve? $n = 15 \times 26$ $= 390$
15 gallons of ice cream will serve 390 children.

9. There were 12 tables in the cafeteria. If 16 children sat at each table, how many children could be served at one time?

$r = 12 \times 16$
$= 192$

192 children could be served at one time.

10. 36 boxes of crayons were ordered for a class. Each box contained 24 crayons. How many crayons were there altogether?

$n = 36 \times 24$
$= 864$

There were 864 crayons altogether.

11. In the parking lot at the ball park there were 24 rows with spaces for 35 cars in each row. How many cars may be parked in this lot?

$s = 24 \times 35$
$= 840$

840 cars may be parked in the lot.

12. It took 191 seconds for the children in Lowell School to leave the building during a fire drill in March. In April the time was 186 seconds. How much longer did it take the children to leave the building in March?

$n = 191 - 186$
$= 5$

It took 5 seconds longer to leave the building.

13. Mrs. Wood made 27 jars of jam. If each jar held 16 ounces, how many ounces did she make?

$s = 27 \times 16$
$= 412$

Mrs. Wood made 432 ounces of jam.

14. There are 53 boys and girls in the morning kindergarten class and 48 in the afternoon class. How many children are in the two classes?

$n = 53 + 48$
$= 101$

There are 101 boys and girls in the two classes.

15. Each of the 32 children in Miss Park's class made 18 name tags for open house. How many name tags did they make?

$r = 32 \times 18$
$= 576$

The children made 576 name tags.

FINDING UNKNOWN FACTORS

Objective: To help children understand the technique of division of large numbers when the unknown number is a factor

"Quotients and remainders" and "exact division are often taught together. The children "divide" 17 by 5 and obtain a quotient 3 and a remainder 2. In symbols,

$$17 = (5 \times 3) + 2.$$

If there is no remainder, the division is "exact". An equivalent statement, used later in this unit, is to say that exact division is the special case of division with remainder when the remainder is zero:

$$15 = (5 \times 3) + 0.$$

In more complicated cases, it is not known whether or not a division is exact until the division is performed.

Quotients and remainders is of the greatest practical importance, but in this unit it is taught after, and not along with, exact division. The reasons for this separation are as follows:

The treatment of arithmetic in Chapters 3, 4, and 6 is based on the concept of a mathematical operation. The child operates on the whole numbers 15 and 5 by multiplication to obtain 75 and by division to obtain 3. In both cases a single whole number results from the operation. Division with remainder is not an operation on whole numbers in this sense. The result of dividing 17 by 5 is two whole numbers 3 and 2, not one. (The point made here is elaborated in the mathematical background for division with remainder.)

The concept of a mathematical operation and its properties is an essential mathematical idea that the child should retain. It is one that will occur repeatedly as he learns more mathematics. It is also the concept which unifies the teaching of the basic algorisms of addition, subtraction, multiplication, and division.

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Exploration:

Multiplication is an operation on two numbers to get a third number.

> Write some examples of mathematical sentences on the chalkboard which indicate multiplication--sentences such as $12 \times 6 = n$, $13 \times 16 = p$, $145 \times 26 = q$, or others that children suggest. Now write on the board such sentences as these: $12 = 3 \times n$, $m \times 15 = 165$, $25 \times s = 450$.

Here are some mathematical sentences. Do these sentences also indicate multiplication? (Yes) What are the factors in each of these sentences? (3 and n, m and 15, 25 and s) What is the product? (12, 165, 450)

How do these differ from those sentences we first wrote? (We know only one factor and the product in the latter. In the first sentences, we know both factors but do not know the product.)

What operation is used to find the unknown factor? (Division) Let us rewrite the last three sentences, using the division symbol.

> Write the sentences:
>
> $12 \div 3 = n$, $165 \div 15 = m$, $450 \div 25 = s$.
>
> Explore possible ways of finding the unknown factor in these sentences. Suggest the multiplication fact $4 \times 3 = 12$ for the first. Note we cannot use the facts in this way to find the others.

We want to find a way which we may use to find m when $m = 165 \div 15$ and s when $s = 450 \div 25$. We might try to find m when $m = 165 \div 15$ by making some guesses and testing our guesses. Make a guess. Test the sentence to find if m is 10. ($15 \times 10 = 150$) Is 10 greater than or less than m? ($10 > m$) Test the sentence to find if m is 20. ($20 \times 10 = 200$) Is 20 greater than or less than m? ($20 < m$) What could be your next guess? (m is smaller than 20 and larger than 10.) We could test to find if m is 15.

> The children should continue this discussion until m and s are determined. Should they start with different guesses, follow the discussion in a similar way.

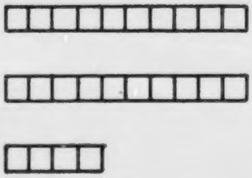
It took us some time to guess the numbers that m and s represent. It would be better if we had a way which could be used with these numbers and with larger numbers that didn't take so much guessing.

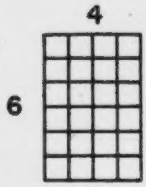
To learn to divide numbers like these, let us begin with some easy division exercises. Learning to do them will help us with more difficult division exercises.

You may take a division sentence as $24 \div 4 = n$. Observe that it indicates we might have an array with 24 elements. We know that it has 4 columns (or 4 elements in each row).

We sometimes associate this kind of sentence, $24 \div 4 = n$, with missing information about the number of rows. We sometimes associate a sentence like $24 \div n = 4$ with missing information about the number of columns. You may wish to delay this idea until later and proceed only with sentences of the form $24 \div 4 = n$ or $n \times 4 = 24$. In either situation, we find the missing factor by dividing the product by the other factor.

Using square-shaped cards, counters, markers, or other similar movable materials, note first the arrangement of 24 as 2 tens and 4 ones. Then rearrange to form an array of 4 columns, with the number of rows to be determined. Children should have similar materials for the same purpose.





$$24 = 6 \times 4$$

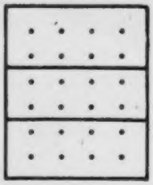
For $24 \div 4 = n$, we can say that $\underline{6} \times 4 = 24$ or $24 \div 4 = \underline{6}$. We can also write

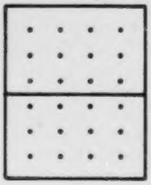
$$\begin{array}{r} 6 \\ 4\overline{)24} \end{array}.$$

You may wish to have children make other arrangements suggested by each of these sentences.

$$\begin{array}{ll} 24 \div 8 = n & 24 \div n = 8 \\ 24 \div 6 = n & 24 \div n = 6 \\ 24 \div 3 = n & 24 \div n = 3 \end{array}$$

Ask how we can write about this. Note the array of 24 elements.


A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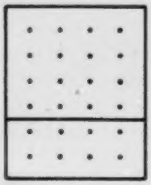
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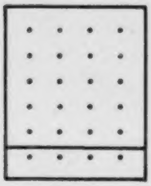

D

We can think of the first array (A) as

$$\begin{aligned} 24 \div 4 &= (8 \div 4) + (8 \div 4) + (8 \div 4) \\ &= \quad 2 \quad + \quad 2 \quad + \quad 2 \\ &= 6 \end{aligned}$$

In B,

$$\begin{aligned} 24 \div 4 &= (12 \div 4) + (12 \div 4) \\ &= \quad 3 \quad + \quad 3 \\ &= 6 \end{aligned}$$

In C,

$$\begin{aligned} 24 \div 4 &= (16 \div 4) + (8 \div 4) \\ &= \quad 4 \quad + \quad 2 \\ &= 6 \end{aligned}$$

In D,

$$\begin{aligned} 24 \div 4 &= (20 \div 4) + (4 \div 4) \\ &= \quad 5 \quad + \quad 1 \\ &= 6 \end{aligned}$$

Recall similar work that was done in Chapter 4.

Note that we can think of separating 24 dots into smaller groups. Each group should be arranged so that there are 4 elements in each row. We can think of expressing the product using multiples of the known factor as addends. For example, 8, 12, 16, 20 are multiples of 4.

Try other examples using the multiplication facts and our knowledge of 10's, 100's, and multiples of 10's and 100's as factors. Some examples:

$$\begin{aligned} 75 \div 5 &= (50 \div 5) + (25 \div 5) \\ &= \quad 10 \quad + \quad 5 \\ &= 15 \end{aligned}$$

$$\begin{aligned} 636 \div 6 &= (600 \div 6) + (30 \div 6) + (6 \div 6) \\ &= \quad 100 \quad + \quad 5 \quad + \quad 1 \\ &= 106 \end{aligned}$$

(It may take several days to develop these ideas.

FINDING UNKNOWN FACTORS

Try to find the unknown factors in these sentences. Use multiplication facts to help you.

A		B	
4 × n = 36	(4 × 9 = 36)	4 × n = 360	(4 × 90 = 360)
n × 8 = 64	(8 × 8 = 64)	n × 8 = 640	(80 × 8 = 640)
n × 7 = 42	(6 × 7 = 42)	n × 7 = 420	(60 × 7 = 420)
2 × n = 12	(2 × 6 = 12)	2 × n = 1200	(2 × 600 = 1200)
5 × n = 30	(5 × 6 = 30)	5 × n = 3000	(5 × 600 = 3000)

What multiplication fact did you use for each?

In what way did finding the unknown factors in A help you to find the factors in B?

(The basic fact used in A helped me to know which multiple of 10 or 100 to use in B.)

How can each of the sentences in A and B be rewritten using the division symbol?

A	B
36 ÷ 4 = n	360 ÷ 4 = n
64 ÷ 8 = n	640 ÷ 8 = n
42 ÷ 7 = n	420 ÷ 7 = n
12 ÷ 2 = n	1200 ÷ 2 = n
30 ÷ 5 = n	3000 ÷ 5 = n

Many problems are solved by dividing one number by another. Here is an example.

Paul has 52 stamps. He can put 4 stamps in one row in his book. How many rows will he need if he puts all 52 stamps in his book?

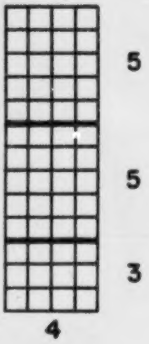
We want to find the number of rows of stamps.

There are 4 stamps in each row.

There are 52 stamps in all.

Is the mathematical sentence for this problem $n \times 4 = 52$, where n represents the number of rows? (yes)

Think of an array.



The stamps in his book might be arranged as in the picture at the right. The way the array is separated shows that

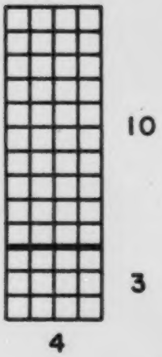
$$52 = 20 + 20 + 12.$$

We write:

$$\begin{aligned} 52 \div 4 &= (20 + 20 + 12) \div 4 \\ &= (20 \div 4) + (20 \div 4) + (12 \div 4) \\ &= 5 + 5 + 3 \\ &= 13. \end{aligned}$$

There are 13 rows of stamps.

The stamps in his book might also be arranged as in this picture at the right.



The way the array is separated shows that $52 = 40 + 12$.

We write:

$$\begin{aligned} 52 \div 4 &= (40 + 12) \div 4 \\ &= (40 \div 4) + (12 \div 4) \\ &= 10 + 3 \\ &= 13. \end{aligned}$$

The number 52 can be renamed so that each addend is a multiple of 4. These numbers are multiples of 4:

4, 8, 16, 20, ...

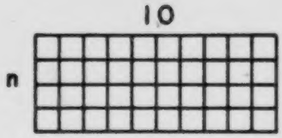
Can you name some others?

Try some other ways of renaming 52.

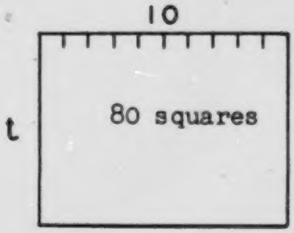
Exercise Set 13

Find the unknown number in each of these exercises.

1.



2.



$n \times 10 = 40$

n = (4)

$t \times 10 = 80$

t = (8)

3. $1200 \div 3 = n$
(n = 400)

4. $810 \div 9 = t$
(t = 90)

5. $p \times 7 = 35$
(p = 5)

6. $420 \div 6 = r$
(r = 70)

7. $q = 640 \div 8$
(q = 80)

8. $y = 770 \div 7$
(y = 110)

Exercise Set 14

Rename each product using multiples of the known factor as addends.

Example 1:

$n = 24 \div 2$

$= (20 + 4) \div 2$

$= (20 \div 2) + (4 \div 2)$

$= 10 + 2$

$= 12$

Example 2:

$r = 393 \div 3$

$= (300 + 90 + 3) \div 3$

$= (300 \div 3) + (90 \div 3) + (3 \div 3)$

$= 100 + 30 + 1$

$= 131$

1. $48 \div 4 = t$ $(t = 12)$
2. $68 \div 2 = s$ $(s = 34)$
3. $96 \div 3 = n$ $(n = 32)$
4. $64 \div 2 = s$ $(s = 32)$
5. $48 \div 2 = m$ $(m = 24)$
6. $42 \div 2 = k$ $(k = 21)$
7. $h = 88 \div 4$ $(h = 22)$
8. $n = 55 \div 5$ $(n = 11)$
9. $75 \div 5 = m$ $(m = 15)$
10. $63 \div 3 = t$ $(t = 21)$
11. $96 \div 6 = r$ $(r = 16)$
12. $91 \div 7 = s$ $(s = 13)$
13. $112 \div 8 = t$ $(t = 14)$
14. $217 \div 7 = w$ $(w = 31)$
15. $333 \div 9 = m$ $(m = 37)$
16. $400 \div 5 = n$ $(n = 80)$
17. $639 \div 3 = k$ $(k = 213)$
18. $420 \div 4 = m$ $(m = 105)$
19. $770 \div 7 = p$ $(p = 110)$
20. $630 \div 6 = t$ $(t = 105)$

Exercise Set 15

Write mathematical sentences to help you solve the following problems. Solve them. Write an answer sentence for each problem.

1. At the air show 40 planes flew in formation. There were 10 rows. How many planes were in each row? $40 = 10 \times n$ $n = 4$
   There were 4 planes in each row.

2. A class of 36 children was divided equally into 3 committees to plan a party. How many children were on each committee? $36 = 3 \times t$ $t = 12$ There were 12 children on each committee

3. The grocer put 60 carrots in bunches of 5. How many bunches did he make? $60 = 5 \times r$ $r = 12$ The grocer made 12 bunches of carrots.

4. How many days are there in 7 weeks?
   $s = 7 \times 7$
   $s = 49$ There are 49 days in 7 weeks.

5. Bob received an allowance of 50¢. Jim's allowance was 75¢. How much more money does Jim receive than Bob?
   $75 - 50 = n$
   $n = 25$ Jim receives 25¢ more than Bob.

6. 50¢ is how many times as much money as 5¢?
   $50 = n \times 5$
   $n = 10$ 50¢ is 10 times as much money as 5¢.

7. The 80 men in the marching band were divided into 8 rows. How many men were in each row?
   $8 \times t = 80$
   $t = 10$ There were 10 men in each row.

8. A jet plane carried 42 passengers in one section and 102 passengers in another section. How many passengers were aboard the jet? $n = 42 + 102$ $n = 144$ There were 144 passengers aboard the jet.

A WAY OF DIVIDING TWO NUMBERS

Objective: To develop a division algorism using as background the idea that the product may be expressed as multiples of the known factor

Materials: Arrays appropriate for following discussion

In this lesson review with the children what they already know about division. Help them think of a way in which they can express the product as a sum of multiples of the known factor as they work.

You may wish to use arrays as you discuss such sentences as $24 \div 4 = n$, for example. Discussion might develop like this:

Suppose we record what happens as we are making an array with 4 cards (or whatever materials are being used) in one row. Here is another way of recording our thoughts.

Suppose we arrange 4 rows of 4.
We have a 4 by 4 array with 16 cards.
We record that we have made 4 rows.
We also write that we have used 16 of the 24 cards.
(At each step note how we record this.)
We see that we have 8 cards left.
We make 2 more rows of 4.
We write this. Also, we write the number of cards needed to make 2 rows of 4.

We have used all the cards. We see from our record that we have 4 rows and 2 rows.
$4 + 2 = 6$

Form I

$$\begin{array}{r} 6 \\ \overline{2} \\ 4 \\ 4\overline{)24} \\ -16 \\ \hline 8 \\ -8 \\ \hline 0 \end{array}$$

Form II

$$\begin{array}{r|l} 4\overline{)24} & \\ -16 & 4 \\ \hline 8 & \\ -8 & 2 \\ \hline 0 & 6 \end{array}$$

Either form may be used.

In dividing 24 by 4 the pupils should recognize that this is equivalent to finding the unknown factor in $4 \times n = 24$; and they will know that the unknown factor is 6 from their knowledge of the multiplication facts. Hence, the rewriting of 24 as the sum of multiples of 4 is not necessary to finding the factor 6. But we are preparing the way for the algorism for division in which the unknown factor is not immediately evident from knowledge of the multiplication facts.

Try other examples, relating the record of the arrangement of an array where only the number of rows or the number of columns is given.

For example, for

$$72 \div n = 7$$

we have 72 elements to be placed in 4 rows. We want to find the number of elements in each row, or the number of columns.

For 72 ÷ 4 = n, there may be several ways of thinking. Here are a few, using both forms.

I.

```
4)72 |
 -40 | 10
  32 |
 -20 |  5
  12 |
 -12 |  3
   0 | 18
```

```
4)72 |
 -32 |  8
  40 |
 -40 | 10
   0 | 18
```

```
4)72 |
 -40 | 10
  32 |
 -32 |  8
   0 | 18
```

II.

```
  18
   5
   3
  10
4)72
 -40
  32
 -12
  20
 -20
   0
```

```
  18
   8
  10
4)72
  40
  32
 -32
   0
```

```
  18
   3
   5
   5
   5
4)72
  20
  52
  20
  32
  20
  12
  12
   0
```

Note how 72 has been expressed as a sum.

$$\begin{aligned} 72 \div 4 &= (40 + 20 + 12) \div 4 \\ &= (32 + 40) \div 4 \\ &= (40 + 32) \div 4 \end{aligned}$$

We are not interested yet in a short-cut form.

A WAY OF DIVIDING TWO NUMBERS

We have learned that we can rename the product and divide to find the unknown factor.

$$\begin{aligned} n &= 225 \div 9 \\ &= (180 + 45) \div 9 \\ &= (180 \div 9) + (45 \div 9) \\ &= \quad 20 \quad + \quad 5 \\ &= 25 \end{aligned}$$

Here is another way to show division.

Mathematical Sentence:

$$n = 225 \div 9$$

Form I:		Form II:	
$\underline{25}$			
5			
20			
$9\,\overline{)225}$		$9\,\overline{)225}$	
$\underline{-180}$	← (20 × 9) →	$\underline{-180}$	20
45		45	
$\underline{-45}$	← (5 × 9) →	$\underline{-45}$	$\underline{5}$
0		0	25

You may use either form shown above.

Now we know

$$\underline{25} \times 9 = 225 \qquad \text{or} \qquad 225 \div 9 = \underline{25}.$$

Exercise Set 16

Use Form I or Form II to find the unknown number in each sentence.

1. $45 \div 3 = n$
$(n = 15)$

2. $76 \div 4 = n$
$(n = 19)$

3. $84 \div 3 = n$
$(n = 28)$

4. $96 \div 8 = n$
$(n = 12)$

5. $72 \div 3 = n$
$(n = 24)$

6. $69 \div 3 = n$
$(n = 23)$

7. $84 \div 7 = n$
$(n = 12)$

8. $60 \div 12 = n$
$(n = 5)$

9. $96 \div 3 = n$
$(n = 32)$

10. $132 \div 6 = n$
$(n = 22)$

11. $444 \div 6 = n$
$(n = 74)$

12. $m = 207 \div 9$
$(m = 23)$

13. $325 \div 5 = p$
$(p = 65)$

14. $376 \div 8 = q$
$(q = 47)$

MORE ABOUT DIVIDING TWO NUMBERS

Objectives: To learn how to divide when products are larger, using multiples of the known factor involving tens and hundreds

To learn to become more skillful in renaming the product

Exploration:

> You may wish to develop this lesson by noting different ways one of the examples has been worked. Follow the form your children are using.

Here are some different ways we found n in 444 ÷ 6 = n.

(a) $$\begin{array}{r} \underline{74} \\ 4 \\ 20 \\ 50 \\ 6\,\overline{)444} \\ \underline{300} \\ 144 \\ \underline{120} \\ 24 \\ \underline{24} \\ 0 \end{array}$$

(b) $$\begin{array}{r} \underline{74} \\ 4 \\ 70 \\ 6\,\overline{)444} \\ \underline{420} \\ 24 \\ \underline{24} \\ 0 \end{array}$$

(c) $$\begin{array}{r} \underline{74} \\ 4 \\ 10 \\ 60 \\ 6\,\overline{)444} \\ \underline{360} \\ 84 \\ \underline{60} \\ 24 \\ \underline{24} \\ 0 \end{array}$$

(a) $$\begin{array}{r|r} 6\,\overline{)444} & \\ \underline{300} & 50 \\ 144 & \\ \underline{120} & 20 \\ 24 & \\ 24 & 4 \\ \hline 0 & 74 \end{array}$$

(b) $$\begin{array}{r|r} 6\,\overline{)444} & \\ \underline{420} & 70 \\ 24 & \\ 24 & 4 \\ \hline 0 & 74 \end{array}$$

(c) $$\begin{array}{r|r} 6\,\overline{)444} & \\ \underline{360} & 60 \\ 84 & \\ \underline{60} & 10 \\ 24 & \\ 24 & 4 \\ \hline 0 & 74 \end{array}$$

Is the missing factor the same in each? (Yes, it is 74.)

How was 444 renamed in (a)? Point out the addends as you rename 444. (First we used 444 = 300 + 144. Then we used 444 = 300 + 120 + 24.) Then what did you do? (We divided 300 by 6, 120 by 6, and 24 by 6.) What were the results? (The results were 50, 20, and 4. These were added to get the unknown factor, 74.)

How was 444 renamed in (b)? (444 = 420 + 24.) Then what did you do? (We divided 420 by 6 and 24 by 6.) What were the results? (The results were 70 and 4. These were added to get the unknown factor, 74.)

How was 444 renamed in (c)? (First, 444 = 360 + 84, then 444 = 360 + 60 + 24.) Then what did you do? (We divided 360 by 6, 60 by 6, and 24 by 6.) What were the results? (The results were 60, 10, and 4. These were added to get the factor, 74.)

Note that these ways of finding 74 are all good. However, we will find our work easier if we can think in the largest multiples of the known factor in 10's or 100's, etc.

Work some problems together, such as

528 ÷ 8 = n

```
8)528 |
  480 | 60
  ---
   48 |
   48 |  6
  ----+---
    0 | 66
```

100 × 8 = 800 and this is greater than 528.

Find the products of 8 and some multiples of 10:

10 × 8 = 80
20 × 8 = 160
50 × 8 = 400
60 × 8 = 480
70 × 8 = 560

```
   66
    6
   60
8)528
  480
  ---
   48
   48
  ---
    0
```

Which product is best to use if we select the largest one that is no more than 528? (480)

You may wish to try other examples before children do those in Exercise Set 17 independently.

MORE ABOUT DIVIDING TWO NUMBERS

You have found that there are several ways to rename a product when you divide. Here are some ways that you may have used to find what number times 6 is 444. $n \times 6 = 444$. Using division we write: $444 \div 6 = n$. You may use either of these forms.

Form I.

```
             74               74
              4                4                74
             20               10                 4
             50               60                70
(a)  6 )444      (b)  6 )444      (c)  6 )444
            300              360               420
            144               84                24
            120               60                24
             24               24                 0
             24               24
              0                0
```

Form II.

```
(a)  6 )444 |      (b)  6 )444 |      (c)  6 )444 |
        300 | 50           360 | 60           420 | 70
        144 |               84 |               24 |
        120 | 20            60 | 10            24 |  4
         24 |               24 |           -------+----
         24 |  4            24 |  4             0 | 74
     -------+----      -------+----
          0 | 74             0 | 74
```

In which one has 444 been renamed as (300 + 120 + 24)? *(a)*

In which one has 444 been renamed as (360 + 60 + 24)? *(b)*

In which one has 444 been renamed as (420 + 24)? *(c)*

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Exercise Set 17

Divide.

1. $3\overset{(83)}{\overline{)249}}$

2. $4\overset{(71)}{\overline{)284}}$

3. $8\overset{(92)}{\overline{)736}}$

4. $5\overset{(73)}{\overline{)365}}$

5. $6\overset{(65)}{\overline{)390}}$

6. $7\overset{(74)}{\overline{)518}}$

7. $7\overset{(56)}{\overline{)392}}$

8. $6\overset{(63)}{\overline{)378}}$

9. $4\overset{(46)}{\overline{)184}}$

10. $3\overset{(84)}{\overline{)252}}$

11. $9\overset{(38)}{\overline{)342}}$

12. $8\overset{(83)}{\overline{)664}}$

13. $5\overset{(90)}{\overline{)450}}$

14. $3\overset{(97)}{\overline{)291}}$

15. $7\overset{(49)}{\overline{)343}}$

16. $9\overset{(79)}{\overline{)711}}$

17. $6\overset{(99)}{\overline{)594}}$

18. $7\overset{(97)}{\overline{)679}}$

19. $8\overset{(88)}{\overline{)704}}$

20. $9\overset{(89)}{\overline{)801}}$

USING DIVISION IN PROBLEM SOLVING

Exploration:

> Before pupils begin work on Exercise Set 18, work with them in a class discussion so that they have opportunities to read and understand problems which are solved by dividing two numbers.
>
> You might begin by having the pupils recall what they know about solving "word problems". Important ideas to be discussed are: 1. Read the problem carefully. 2. Note the question that is asked. 3. Look for information related to the question. (It may be necessary to reread the problem.) 4. Write a mathematical sentence which uses this information to answer the question. 5. Study the mathematical sentence to determine what operation to use. 6. Then compute. 7. Write an answer sentence which answers the question asked in the problem.
>
> After pupils have recalled the preceding ideas about problem solving, write this problem on the board:

A fourth grade class, which has 24 children, visited a museum. Six cars were used to take the children there. If the same number of children rode in each car, how many children were in each car?

What question is asked? (How many children were in each car.)

What information is given? (There are 24 children in the class. Six cars were used to take the children to the museum.)

What mathematical sentence can you write which uses this information to answer the question? ($6 \times n = 24$ or $24 \div 6 = n$) Find the answer to the question. What is your answer sentence? (There were 4 children in each car.)

You found the number of children in each car. We can divide when we want to know how many are in each of several equal sets.

Can you solve this problem?

> Write it on the board.

At the museum the 24 children were put into groups of 8. How many groups of children were there?

What question is asked? (How many groups of children were there?)

What information did you find to help answer this question? (There are 24 children. The children are put into groups of 8.)

What mathematical sentence can you write which uses this information to answer the question? ($8 \times n = 24$ or $24 \div 8 = n$) Find the answer to the question. What is your answer sentence? (There were 3 groups of children.)

You found that there were 3 groups of children. Each of the groups had the same number of children in it. We can divide when we want to know how many groups of the same size there are.

What are the two kinds of problems in which we can divide? (1. We can divide if we know the number of sets with an equal number in each group and we want to know how many are in each set. 2. We can divide if we know how many are in each set and we want to know how many sets there are.)

In the preceding problems, the children could use multiplication facts to get the answer. Now use the following examples in which children can use division algorisms.

Now let's try an example in which we work with larger numbers.

Write this problem on the board.

Rangers in a park counted 420 deer. They need to feed the deer during the winter. The rangers decided to put the deer into 3 groups with an equal number in each group. How many deer were in each group?

Follow the same procedure as was done with the other examples.

USING DIVISION IN PROBLEM SOLVING

There are 108 fruit trees in an orchard. There are 9 rows of trees with the same number of trees in each row. How can you find the number of trees in each row?

The information in the problem is:

There are 108 trees.

There are 9 rows.

Each row has the same number of trees.

The question we want to answer is:

How many trees are there in each row?

Let us form a mathematical sentence to show how the bits of information in the problem are related. Let n represent the number of trees in each row.

$$9 \times n = 108, \quad \underline{\text{or}} \quad n = 108 \div 9.$$

In the mathematical sentence, 108 is the product, 9 is the known factor, and n is the unknown factor. We can find n by dividing 108 by 9. Your answer should be 12 so that

$$9 \times 12 = 108.$$

We now write an answer sentence:

There are 12 trees in each row.

In this problem about the trees, there are 108 trees in the set. The set of 108 trees is divided into 9 sets with the same number in each group. You found the number of trees in each of the 9 sets. This number was 12. You used division to find the number of trees in each set.

Now let us think of another problem. Suppose there are 822 dogs in a large dog show. An official tells us that there are cocker spaniels, poodles, collies, Irish setters, boxers, and German shepherds. Also, he tells us that in each breed there is the same number of dogs. How many dogs of each breed are in the show?

The information in the problem is:

There are 822 dogs in the show.

There are 6 breeds of dogs in the show.

Each breed has the same number of dogs.

The question we want to answer is:

How many dogs are there of each breed?

Let us form a mathematical sentence. Let n represent the number of each breed of dog. $6 \times n = 822$, or $n = 822 \div 6$.

In the mathematical sentence, 822 is the product, 6 is the known factor, and n is the unknown factor. We can find n by dividing 822 by 6. We show the division in either one of these ways.

```
        137
          7
         30
        100
     6 )822          6 )822 |
        600             600 | 100
        ---             --- |
        222             222 |
        180             180 |  30
        ---             --- |
         42              42 |
         42              42 |   7
        ---             ----+----
          0               0 | 137
```

The answer sentence is: There are 137 dogs of each breed in the show.

In this problem about dogs, there are 822 dogs in the set. There are 6 sets with the same number in each group. We found the number of dogs in each set by dividing 822 by 6.

Exercise Set 18

Write mathematical sentences to help you solve the following problems. Solve them. Write an answer sentence for each problem.

1. The Jackson family is planning a 510 mile trip. If they have 5 days to travel, about how many miles must they travel each day?
   $5 \times n = 510$
   $n = 102$
   or $n = 510 \div 5$
   The Jackson family must travel 102 miles each day.

2. Seven jets left the airport one day. Each had 128 passengers aboard. How many people left the airport by jet that day?
   $n = 7 \times 128$
   $n = 896$
   896 people left the airport by jet that day.

3. In Cub Scouts, John made a collection of 144 small shells. He put the same number in each of 6 boxes. How many did he put in each box?
   $n \times 6 = 144$
   $n = 24$
   or $144 \div 6 = n$
   John put 24 shells in each box.

4. Fran collected 126 leaves for a project in school. She mounted them on 9 large posters. How many leaves did she mount on each poster?
   $n \times 9 = 126$
   OR
   $126 \div 9 = n$
   $n = 14$
   Fran mounted 14 leaves on each poster.

5. There are 189 Boy Scouts in 9 troops. If each troop has the same number of members, how many boys are in each troop?
   $n \times 9 = 189$
   OR
   $189 \div 9 = n$
   $n = 21$
   There are 21 boys in each troop.

6. Peggy's mother baked 186 cookies for a picnic. She packed the same number in each of three boxes. How many did she pack in each box?
   $n \times 3 = 186$
   OR $186 \div 3 = n$
   $n = 62$
   She packed 62 cookies in each box.

7. Dick and Tom offered to make tickets for the puppet show. They made 139 tickets on Tuesday, 125 on Wednesday, and 127 on Thursday. How many tickets did the boys make together? $n = 139 + 125 + 127$
$n = 391$ The boys made 391 ticket.

8. The restaurant had 2 dining rooms. One held 220 people, the other had room for 175 people. How many more people could eat in one dining room than in the other?
$n = 220 - 175$
$n = 45$ 45 more people could eat in one dining room than in the other.

9. If 27 visitors are taken through a state capitol building in one group, how many visitors are taken through in 13 groups? $n = 13 \times 27$ 351 people were taken through the capitol building.
$n = 351$

10. If one case of canned soup weighs 24 pounds, how much will 48 cases weigh?
$n = 24 \times 48$ 48 cases of canned soup will weigh 1,152 pounds.
$n = 1152$

11. A committee of 7 pupils collected 455 rocks while working on a class project. If each pupil collected the same number of rocks, how many rocks did each pupil find? $7 \times s = 455$ Each pupil found 65 rocks.
OR
$s = 455 \div 7$
$s = 65$

12. There are 9 boys in our Cub Scout den. The boys collected 477 toys during their yearly toy drive. If each boy collected the same number of toys, how many toys did each boy collect?
$9 \times t = 477$ Each boy collected 53 toys.
OR
$t = 477 \div 9$
$t = 53$

BECOMING MORE SKILLFUL IN DIVIDING NUMBERS

Objective: To provide more practice in division and at the same time learn how to divide when products are larger and the unknown factor is larger than 100

Exploration:

You may choose to break the exploration of techniques of multiplication and division at some convenient point and review or develop some geometric ideas. This will give the children time to absorb the properties and techniques which have been developed.

The aim of this unit is to help the child learn the division technique which requires a renaming of the product as a sum and dividing each addend of the sum to find the missing factor.

The children should improve their efficiency in renaming a product as a sum for dividing; but at this stage they should not all be expected to achieve the same level of skill. Here are examples of what can be expected, perhaps in order of preference.

$$\begin{array}{r} 89 \\ \hline 9 \\ 80 \\ 4\,\overline{)356} \\ 320 \\ \hline 36 \\ 36 \\ \hline \end{array} \qquad \begin{array}{r} 89 \\ \hline 11 \\ 8 \\ 70 \\ 4\,\overline{)356} \\ 280 \\ \hline 76 \\ 32 \\ \hline 44 \\ 44 \\ \hline \end{array} \qquad \begin{array}{r} 89 \\ \hline 9 \\ 10 \\ 10 \\ 60 \\ 4\,\overline{)356} \\ 240 \\ \hline 116 \\ 40 \\ \hline 76 \\ 40 \\ \hline 36 \\ 36 \\ \hline \end{array} \qquad \begin{array}{r} 89 \\ \hline 9 \\ 60 \\ 20 \\ 4\,\overline{)356} \\ 80 \\ \hline 276 \\ 240 \\ \hline 36 \\ 36 \\ \hline \end{array}$$

The following is suggested as a guide to the discussion in learning how to divide when the products are larger than 100 times the unknown factor. Show work on the chalkboard.

What is the first step in dividing 472 by 4? (We rename 472 as a sum so we can divide each addend by the known factor.) (We know 4 × 100 = 400. The first addend of the name for 472 is 400.) Why did we choose 400 instead of a smaller addend? (We chose 400 because it is the smallest multiple of 100 that is no more than 472.) The two addends used in renaming 472 are 400 and 72.

$$\begin{array}{r} 100 \\ 4\,\overline{)472} \\ \underline{400} \\ 72 \end{array} \qquad\qquad \begin{array}{r|l} 4\,\overline{)472} & \\ \underline{400} & 100 \\ 72 & \end{array}$$

Is the division complete? (No, we have divided only 400 by 4. We have 72 to be divided by 4.) How can the 72 be renamed as a sum? (It can be renamed as 40 + 32, 36 + 36, 28 + 44, etc.) All of these could be used. Which of these names is best to use? (40 + 32) Must I rename 72 at all? (No, not if I know 72 ÷ 4 = 18.) Divide each by 4. (40 ÷ 4 = 10. 32 ÷ 4 = 8) Show this division on the chalkboard, and show the missing addend.

$$\begin{array}{r} \underline{118} \\ 8 \\ 10 \\ 100 \\ 4\,\overline{)472} \\ \underline{400} \\ 72 \\ \underline{40} \\ 32 \\ \underline{32} \\ 0 \end{array} \qquad \text{or} \qquad \begin{array}{r|r} 4\,\overline{)472} & \\ \underline{400} & 100 \\ 72 & \\ \underline{40} & 10 \\ 32 & \\ 32 & 8 \\ \hline 0 & 118 \end{array}$$

How was 472 named? (400 + 40 + 32) (Point out these addends in the work on the chalkboard. Point out where 400 ÷ 4 is written. Point out where 40 ÷ 4 and 32 ÷ 4 are written. Point out where 472 ÷ 4 is written.)

Let us solve another example together.

$$3\overline{)867}$$

Will the missing factor be at least 100? (Yes, $3 \times 100 = 300$.) Will the missing factor be at least 200? (Yes. $3 \times 200 = 600$.) Will it be at least 300? (No. $3 \times 300 = 900$. 300 is too large.) We shall use 600 as one addend in renaming 867 as a sum. What is the other addend? (267, since $867 = 600 + 267$.)

$$\begin{array}{r} 200 \\ 3\overline{)867} \\ -\underline{600} \\ 267 \end{array} \qquad\qquad \begin{array}{r|l} 3\overline{)867} & \\ -\ 600 & 200 \\ \cline{1-1} 267 & \end{array}$$

Which addend is still to be divided by 3? (267) Can we rename 267 as a sum of addends so that one addend is a multiple of 10 and also a multiple of 3? What are some numbers that are multiples of 3? (Some of these are 30, 60, 90, ...) What is the largest one of these that is less than 267? (240) We can use 240 as one of these that is less than 267? (240) We can use 240 as one of the addends in renaming 267. Is the other addend 27? (Yes) Can we divide 27 by 3? (Yes) Have we renamed 867 as $600 + 240 + 27$? (Yes)

Now we can show the steps in the division process in either of these two ways.

Form I

$$\begin{array}{r} \underline{289} \\ 9 \\ 80 \\ 200 \\ 3\overline{)867} \\ \underline{600} \\ 267 \\ \underline{240} \\ 27 \\ \underline{27} \\ 0 \end{array} \qquad\qquad \begin{array}{r|r} 3\overline{)867} & \\ 600 & 200 \\ \cline{1-1} 267 & \\ 240 & 80 \\ \cline{1-1} 27 & \\ 27 & 9 \\ \hline 0 & 289 \end{array}$$

$$867 \div 3 = 289$$

Does this mean that $289 \times 3 = 867$? (Yes) Multiply 289 by 3 and see if the product is 867. We can always check our work by multiplying.

Let us study these examples to decide on a way to begin a division example.

(a) $5\,\overline{)1620}$ (b) $8\,\overline{)9280}$ (c) $4\,\overline{)3124}$

Can you tell by studying these examples how many digits are in the numeral which names the unknown factor? How many digits will there be in the result of exercise (a)? (Three, because $5 \times 100 < 1620$, but $5 \times 1000 > 1620$; so the unknown factor will be greater than 100 and less than 1000.) How many digits will there be in the result of exercise (b)? (Four, because $8 \times 1000 < 9820$, but $8 \times 10{,}000 > 9280$; so the unknown factor will be greater than 1000 and less than 10,000.) How many digits will there be in the result of exercise (c)? (Three, because $4 \times 100 < 3124$ and $4 \times 1000 > 3124$; so the unknown factor will be greater than 100 and less than 1000.)

Knowing how many digits there are in the numeral which names the unknown factor can help us in beginning a division exercise. In (a) you know the unknown factor must be between 100 and 1000. What multiples of 100 can you multiply by 5 and get a number no more than 1620? (100, 200, 300) We can use this to begin.

$$\begin{array}{rr} & 300 \\ 5 & \overline{)1620} \\ & \underline{1500} \\ & 120 \end{array} \qquad \text{or} \qquad \begin{array}{rr|l} 5 & \overline{)1620} & \\ & \underline{1500} & 300 \\ & 120 & \end{array}$$

Now have the pupils tell how to complete dividing 1620 by 5.

Then lead them by similar questions to determine the largest multiple of 1000 they can multiply by 8 and get a number no more than 9280 in (b) and, the largest multiple of 100 they can multiply by 4 and get a number no more than 3124 in (c).

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BECOMING MORE SKILLFUL IN DIVIDING NUMBERS

The fourth grade class had 1720 inches of string. They wanted to cut it into pieces, each 8 inches long. How many pieces will they have?

Mathematical sentence: 1720 ÷ 8 = n or n × 8 = 1720

We can work this problem in several ways. Here are three ways.

Form I:

```
(a)   215          (b)   215           (c)
      ---                ---
        5                  5                  215
       60                 10                  ---
      100                100                    5
       50                100                   10
 8 )1720            8 )1720                   200
     400                800             8 )1720
    ----               ----                1600
    1320                920                ----
     800                800                 120
    ----               ----                  80
     520                120                ----
     480                 80                  40
    ----               ----                  40
      40                 40                ----
      40                 40                   0
    ----               ----
       0                  0
```

Is (c) the shortest one of the 3 ways?

There are 215 pieces of string.

__Form__ __II__:

(a)

```
8 )1720 |
    400 |  50
   ---- |
   1320 |
    800 | 100
   ---- |
    520 |
    480 |  60
   ---- |
     40 |
     40 |   5
   -----+----
      0 | 215
```

(b)

```
8 )1720 |
    800 | 100
   ---- |
    920 |
    800 | 100
   ---- |
    120 |
     80 |  10
   ---- |
     40 |
     40 |   5
   -----+----
      0 | 215
```

(c)

```
8 )1720 |
   1600 | 200
   ---- |
    120 |
     80 |  10
   ---- |
     40 |
     40 |   5
   -----+----
      0 | 215
```

Is (c) the shortest of the three ways?

Exercise Set 19

Find the missing factor.

1. $340 \div 4 = n$
($n = 85$)

2. $567 \div 9 = n$
($n = 63$)

3. $1435 \div 5 = n$
($n = 287$)

4. $1056 \div 8 = n$
($n = 132$)

5. $372 \div n = 6$
($n = 62$)

6. $504 \div 7 = t$
($t = 72$)

7. $474 \div m = 6$
($m = 79$)

8. $420 \div 4 = p$
($p = 105$)

9. $369 \div 3 = n$
($n = 123$)

10. $2240 \div 4 = m$
($m = 560$)

11. $5250 \div m = 7$
($m = 750$)

12. $8280 \div 9 = t$
($t = 920$)

13. $3616 \div 8 = n$
($n = 452$)

14. $3560 \div 2 = n$
($n = 1780$)

15. $4362 \div 3 = k$
($k = 1454$)

16. $8960 \div 8 = s$
($s = 1120$)

17. $5761 \div 7 = m$
($m = 823$)

18. $3768 \div 4 = t$
($t = 942$)

19. $9384 \div 6 = p$
($p = 1564$)

20. $9639 \div 9 = s$
($s = 1071$)

Exercise Set 20

Write mathematical sentences to help you solve the following problems. Solve them. Write an answer sentence for each problem.

1. If a plane travels 1675 miles in 5 hours, about how far does it travel in one hour?

   $n = 1675 \div 5$

   $n = 335$ The plane travels 335 miles in one hour.

2. How many doughnuts are there in 17 dozen?

   $n = 17 \times 12$

   $n = 204$ There are 204 doughnuts in 17 dozen.

3. For the school carnival the mothers put 600 pieces of homemade candy into bags. They put 5 pieces of candy in each bag. How many bags did they pack?

   $n = 600 \div 5$ The mothers packed 120 bags of candy.

   $n = 120$

4. 720 ice cream bars were bought by the Dad's Club to treat the children of Baker School. There were 669 children present that day. How many extra bars were there?

   $n = 720 - 669$ There were 51 extra ice cream bars.

   $n = 51$

5. A motorcycle traveled 234 miles on 6 gallons of gas. How far did it travel on one gallon?

   $n = 234 \div 6$ The motorcycle traveled 39 miles on one gallon of gas.

   $n = 39$

6. A market put 1744 onions into bunches of 8 onions each. How many bunches were there?

   $s = 1744 \div 8$ The market made 218 bunches of onions.

   $s = 218$

7. A grocer ordered 726 bottles of soft drinks. They were delivered in cartons that held six bottles each. How many cartons were delivered?

   $t = 726 \div 6$ The grocer received 121 cartons of soft drinks.

   $t = 121$

FINDING QUOTIENTS AND REMAINDERS

Objective: To help children understand the technique of division with remainders and the mathematical sentence which describes this division process $a = (b \times q) + r$ or $a = (q \times b) + r$ where a is the given product, b is the known factor, q is the unknown factor, r is the remainder, and r is less than b

> The operation of division is defined as finding an unknown factor n in a multiplication: $(5 \times n = 40,\ n = 40 \div 5)$. A division like $39 \div 5$ is impossible within the set of whole numbers. Nevertheless, in many situations it is useful to find: a) the largest number smaller than 39 with 5 as a factor (35), b) the corresponding unknown factor (7), and c) the difference between 39 and a) (4). All of this information can be shown in the sentence $39 = (7 \times 5) + 4$.
>
> As soon as the idea of rational numbers is developed, there is no need to distinguish "division" from "finding quotients and remainders." For $39 \div 5 = \frac{39}{5}$, and the content of $39 = (7 \times 5) + 4$ can be shown by the fraction $7\frac{4}{5}$, which is another name for $\frac{39}{5}$.
>
> At present, we are using just the whole numbers. Consequently, a difficulty arises in the development of division with remainder. The children know that a mathematical operation on whole numbers starts with a pair of whole numbers and results in a third whole number, $(10 + 5 = 15,\ 10 - 5 = 5,\ 10 \times 5 = 50,\ 10 \div 5 = 2)$. Therefore, it will not do to say that the mathematical sentence, $39 = (7 \times 5) + 4$ or $39 = (5 \times q) + r$, represents an operation on the pair of numbers 39 and 5, for it results in two numbers, 7 and 4, and not in one. Consequently, it will be advisable to **encourage** the expression, "Finding Quotients and Remainders" for situations in which division is not possible in the set of whole numbers. In this manner, the child will still associate "finding an unknown factor" with the division operation. "Finding quotients and remainders" will be associated with situations as "135 divided by 12" in

which we must find numbers for q and r in the mathematical sentence

$$135 = (q \times 12) + r.$$

In summary, given any two whole numbers, for example 96 and 6, if there is a whole number n such that $6 \times n = 96$ we call 96 the product, 6 the known factor, and n the unknown factor. The unknown factor n is found by dividing 96 by 6, and we write $n = 96 \div 6$. This is the operation of division.

Now if, for example, we consider the number 96 and 6, there is no whole number n such that $6 \times n = 96$, but there is always a whole number q and a whole number r, $(r < 6)$, so that

$$98 = (q \times 6) + r$$

where $r < 6$. Finding the numbers q and r is "finding quotient q and remainder r." Clearly, q is 16 and r is 2 and

$$98 = (16 \times 6) + 2.$$

Exploration:

We have thought about many division problems, and we know many situations which require division. Here is one for you to solve: "Seventeen boys want to have a relay race with five boys on a team. They line up in rows, five boys in each row. How many teams will there be?"

Figure 16

Explain your drawing. (There are 3 teams of 5 boys each with 2 boys left out of the race.) How many boys will race? (15) How many teams will race? (3)

Is your drawing an array? (No) Why not? (An array must have the same number of elements in each row. There are only 2 dots in the last row.)

Can you see two arrays in your drawing? (I see one 3 by 5 array. We could **describe** the other part as a 1 by 2 array.) What does each of these arrays represent in the problem? (The 3 by 5 array represents the 15 boys who will race. The 1 by 2 array represents the 2 boys who will not race.)

Write a mathematical sentence which shows what we have found. $[17 = (3 \times 5) + (1 \times 2)]$ Each number in this sentence is important. Tell what each number means in the problem. (17 is the number of boys. The boys can form 3 teams of 5 each with 2 boys not in the race.)

What numbers were given in the problem? (17 and 5) What numbers were missing in the problem? (3 and 2)

We may solve problems like this by drawing a picture, separating it into arrays. The result may be described by a mathematical sentence. The result may be used to answer the question in the problem.

If another detailed example is needed, the following may be used.

Let us solve this problem together. "22 boys want to play regulation baseball. How many teams can be organized, and how many boys will be substitutes?"

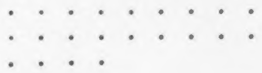


Figure 17

Is this an array? (No, it is two arrays.) Explain the drawing. (There are 2 teams of 9 players and 4 substitutes.) How many boys make up the 2 teams? (18)

Explain the use of each array. (The 2 by 9 array represents the 18 boys who will play. The 1 by 4 array represents the 4 boys who will be substitutes.)

Write a mathematical sentence which shows how the 22 boys are organized. $[22 = (2 \times 9) + 4.]$ The smaller array will always have either 1 row or 1 column. We do not need to write 1×4.

Answer the questions in the problem. (There can be 2 teams with 4 substitutes.)

What numbers were given in the problem? (22 and 9) What numbers were missing in the problem? (2 and 4)

Consider this problem: If the school custodian put 50 chairs into rows of 4 chairs each, how many rows could he make?

Without drawing a picture, can you write a mathematical sentence which shows how the custodian could arrange the 50 chairs into rows of 4 chairs each? [$50 = (12 \times 4) + 2$.] How would the chairs be arranged? (The chairs would be arranged in 12 rows of 4 chairs each with 2 chairs unused.) Write sentences showing several ways to arrange the 50 chairs.

Some possible suggestions:

(a) $50 = (10 \times 5) + 0$
(b) $50 = (9 \times 5) + 5$
(c) $50 = (8 \times 6) + 2$
(d) $50 = (7 \times 7) + 1$
(e) $50 = (11 \times 4) + 6$
(f) $50 = (25 \times 2) + 0$

Each of these sentences shows three things. What are they? (Each sentence shows the number of rows that could be made, the number of chairs in a row, and the number of chairs unused.

Develop a procedure for finding q and r using the same process as for finding the missing or unknown factor:

Form I.

$$\begin{array}{r} \underline{12} \\ \underline{2} \\ 10 \\ 4\,\overline{)50} \\ \underline{40} \\ 10 \\ \underline{8} \\ 2 \end{array}$$

Form II.

$$\begin{array}{r|r} 4\,\overline{)50} & \\ \underline{40} & 10 \\ 10 & \\ \underline{8} & \underline{2} \\ 2 & 12 \end{array}$$

Identify q and r and write the mathematical sentence

$$50 = (12 \times 4) + 2.$$

Similar illustrations may be given for 3 chairs in a row, etc. Note that r is always less than the divisor. (Unless we know that two numbers are a factor of a third, it is improper to write such sentences as $12 \div 5 = n$ unless we want to note that there is no solution if n is to represent a whole number.)

Use other illustrations in this exploration.

FINDING QUOTIENTS AND REMAINDERS

We can use the division process to solve problems like this one:

There are 306 people at the exhibit. There are to be 4 tours. How many people should go in each tour to have about the same number of people in each group?

Mathematical sentence: $306 = (4 \times n) + r$.

You may use either form.

Form I.

$$\begin{array}{r} \underline{76} \\ 6 \\ 70 \\ 4\;\overline{)306} \\ \underline{280} \\ 26 \\ \underline{24} \\ 2 \end{array}$$

Form II.

$$\begin{array}{r|r} 4\;\overline{)306} & \\ \underline{280} & 70 \\ 26 & \\ 24 & 6 \\ \hline 2 & 76 \end{array}$$

$$306 = (76 \times 4) + 2$$

Each group should have 76 people. There are 2 people to join one or two of the groups.

In a mathematical sentence like

$$306 = (76 \times 4) + 2,$$

we say that:

306 is the dividend.
76 is the quotient.
4 is the divisor.
2 is the remainder.

Exercise Set 21

Find the numbers q and r must represent to make each sentence true.

1. $632 = (q \times 9) + r$
   $632 = (\underline{70} \times 9) + \underline{2}$

2. $456 = (q \times 3) + r$
   $456 = (\underline{152} \times 3) + \underline{0}$

3. $1576 = (q \times 5) + r$
   $1576 = (\underline{315} \times 5) + \underline{1}$

4. $1242 = (q \times 8) + r$
   $1242 = (\underline{155} \times 8) + \underline{2}$

5. $943 = (q \times 7) + r$
   $943 = (134 \times 7) + 5$

6. $1210 = (q \times 6) + r$
   $1210 = (201 \times 6) + 4$

7. $421 = (q \times 3) + r$
   $421 = (140 \times 3) + 1$

8. $(q \times 4) + r = 3320$
   $(830 \times 4) + 0 = 3320$

9. $299 = (q \times 7) + r$
   $299 = (42 \times 7) + 5$

10. $151 = (q \times 4) + r$
    $151 = (37 \times 4) + 3$

11. $525 = (q \times 8) + r$
    $525 = (65 \times 8) + 5$

12. $373 = (q \times 5) + r$
    $373 = (74 \times 5) + 3$

Exercise Set 22

Write mathematical sentences to help you solve the following problems. Solve them. Write an answer sentence for each problem.

1. 29 boys want to organize teams of 5 boys for a relay race. How many teams can be organized? How many boys will not race?

$29 = (q \times 5) + r$ There can be 5 teams organized.
$29 = (5 \times 5) + 4$ 4 boys will not race.

2. If the school custodian put 80 chairs into rows of 10 chairs each, how many rows could he make? Would there be any chairs not used?

$80 = (q \times 10) + r$ The custodian could make 8 rows.
$80 = (8 \times 10) + 0$ All chairs would be used.

3. Longfellow School bought 25 new bounce balls. Each of 6 classrooms are to share the balls equally. Any that are left will be kept for next year. How many balls will each room get?

$25 = (q \times 6) + r$ Each room will get 4 new bounce balls.
$25 = (4 \times 6) + 1$ 1 ball will be kept for next year.

4. "Polka for Three" is a dance done in groups of 3. How many groups can be made in a class of 32 children? How many children will not dance?

$32 = (q \times 3) + r$ 10 groups can be made in a class of 32 children.
$32 = (10 \times 3) + 2$ There will be 2 children who will not dance.

5. Mary is asking 30 girls to her party. How many tables must she have if she serves 4 at a table? How many girls will have to sit on the sofa to eat?

$30 = (q \times 4) + r$ Mary must have 7 tables.
$30 = (7 \times 4) + 2$ 2 girls will have to sit on the sofa to eat.

6. 271 reservations were made for a luncheon. How many tables would have to be set if 4 people were to be seated at each table?

$271 = (q \times 4) + r$ 68 tables will be set. There will be 67 full
$271 = (67 \times 4) + 3$ tables and 1 table of 3.

REVIEWING AND EXTENDING

Objective: To help children further their ability to solve problems through the use of mathematical sentences

> The method of problem solving emphasized in Chapter 3 and reemphasized in this unit should be expanded and reinforced by the exploration which follows.
>
> The children should be encouraged to use good judgment as they use the results obtained from the use of the division process to answer the question in the problem.

Exploration:

You have already solved some problems which required the division process. We shall study others with large numbers and consider how we can become better at problem solving.

There are some problems in your book which we shall study. They are in Exercise Set 23.

1. A baker is to pack 1250 cupcakes for a school picnic. He will put 8 in each box. How many boxes shall he order?

2. Each of 15 Girl Scouts sold 24 boxes of cookies. How many boxes were sold?

3. Six sheets of colored paper are needed for a booklet. How many booklets can be made from 500 sheets of colored paper?

4. The Parent Teacher Association of a school had 324 members last year and 296 members this year. How many more memberships are needed to reach last year's record?

Read Problem (1). What information is given in the problem? (There are 1250 cupcakes to be packed 8 to a box.) Write this as a mathematical sentence. ($1250 = q \times 8$.) Will there be a remainder? (We cannot tell.) How do you show this in the sentence? $[1250 = (q \times 8) + r]$ Does the sentence tell you what operation to use to find q? (Yes, we should divide 1250 by 8.)

Write the mathematical sentence on your paper. Then, divide to find q. Answer the question asked in the problem. (He will need 156 boxes to pack all of the cupcakes except 2 and one box for the 2 cupcakes. He will need 157 boxes.)

The work of the children may follow either form.

$$1250 = (q \times 8) + r$$

```
     156
       6
      50
     100
8 )1250          or          8 )1250 |
    800                          800 |  100
    ---                          --- |
    450                          450 |
    400                          400 |   50
    ---                          --- |
     50                           50 |
     48                           48 |    6
    ---                          ----+----
      2                            2 |  156
```

$$1250 = (156 \times 8) + 2$$

The baker should order 157 boxes.

The children should also solve problems (2), (3), and (4) as a group or individually. the following exploration should be made after the children have completed the first four exercises of Exercise 23.

Let us write on the chalkboard the four mathematical sentences you used in the problems so we may study them.

(1) $1250 = (q \times 8) + r$ (2) $m = 15 \times 24$

(3) $500 = (q \times 6) + r$ (4) $296 + n = 324$

Each of the sentences you have written is of a different kind. How did you know what sentences to write? (Answers will vary.)

Why did you write $1250 = (q \times 8) + r$ for the relationship in problem (1)? (The baker had packed 1250 cupcakes in q boxes with 8 to a box. The r was added to take care of any extra cupcakes.)

Why did you write $m = 15 \times 24$ for the relationship in problem (2)? (There were 15 girls who sold 24 boxes each. We thought of a 15 by 24 array.)

Why did you write $500 = (q \times 6) + r$ in problem (3)? (The 500 sheets of paper were made into q booklets of 6 sheets. The r was added to take care of any extra sheets of paper.)

Why did you write $296 + n = 324$ for problem (4)? (The words in the problem told us that 296 members and some new members are the same as 324.)

You have solved four problems by finding the answers to the question asked in the problem. What operation did you use to solve these problems? (We used addition in one, multiplication in one, and division in two.) How did you know what operation to use? (The mathematical sentence told us.)

Look at your work for the two problems in which you used division.

(1) $1250 = (q \times 8) + r$

```
       156
       ---
         6
        50
       100
  8 )1250      or    8 )1250 |
       800                800 | 100
      ----               -----|
       450                450 |
       400                400 |  50
      ----               -----|
        50                 50 |
        48                 48 |   6
      ----               -----+-----
         2                  2 | 156
```

$1250 = (8 \times 156) + 2$

The baker should order 157 boxes.

(3) $500 = (q \times 6) + r$

```
        83
        --
         3
        80
  6 )500      or    6 )500 |
       480              480 |  80
      ----             -----|
        20               20 |
        18               18 |   3
      ----             -----+----
         2                2 |  83
```

$500 = (83 \times 6) + 2$

They can make 83 booklets with 2 extra sheets.

The 157 which you used as an answer in problem (1) does not appear in your work. Why not? (The missing factor is 156 with a remainder of 2. We found that there would be 2 cupcakes left over unless we used another box.) You need to use good judgment in answering questions in problems like this.

The 83 which you used as an answer in problem (3) is in the division work you did. Why did you ignore the remainder?

> (Since it takes 6 sheets for each booklet, only 83 booklets could be made. The 2 sheets of paper left over were not enough for another booklet.)

You were told by the problem how to use the remainder. In solving problems it is a good plan to let the words of the problem guide your work.

Exercise Set 23

1. A baker is to pack 1250 cupcakes for a school picnic. He will put 8 in each box. How many boxes will he order?

$1250 = (q \times 8) + r$ He will order 157 boxes.

$1250 = (156 \times 8) + 2$

2. Each of 15 Girl Scouts sold 24 boxes of cookies. How many boxes were sold?

$n = 15 \times 24$ The Girl Scouts sold 360 boxes of cookies.

$n = 360$

3. Six sheets of colored paper are needed for a booklet. How many booklets can be made from 500 sheets of colored paper?

$500 = (q \times 6) + r$ They can make 83 booklets with 2 extra sheets.

$500 = (83 \times 6) + 2$

4. The Parent Teacher Association of a school had 324 members last year and 296 members this year. How many more memberships are needed to reach last year's record?

$296 + n = 324$ 28 more members are needed.

$n = 28$

5. In a school library there were 23 sets of readers. There were 35 books in each set. How many books were there in the 23 sets?

$t = 23 \times 35$ There were 805 books.

$t = 805$

6. Bill planted 12 rows of tomatoes. There were 15 plants in each row. How many plants did Bill set out?

$s = 12 \times 15$ Bill set out 180 plants.

$s = 180$

7. The children of Miller School are raising money to buy a television set which costs $350. They have collected $179. How much more money do they need?

$350 = 179 + m$ The children need $171 more.

$m = 171$

8. Nancy is making some decorations for a party. She needs 360 white beads, 720 red beads, 180 green beads, and 45 yellow beads. How many beads does she need altogether?

$n = 360 + 720 + 180 + 45$

$n = 1305$

Nancy needs 1305 beads.

9. If a jet travels 408 miles an hour, how far will it travel in 5 hours?

$n = 5 \times 408$

$n = 2040$

The jet travels 2,040 miles in 5 hours.

10. There were 385 tickets to be put in bundles of 8. How many bundles will there be? Will any tickets be left?

$385 = (q \times 8) + r$

$385 = (48 \times 8) + 1$

There will be 48 bundles of ticket with 1 ticket left.

11. On a reading test, Mary read 284 words in 3 minutes. About how many words did she read in one minute?

$284 = (q \times 3) + r$

$284 = (94 \times 3) + 2$

Mary read between 94 and 95 words a minute.

12. A farmer packed 360 boxes of apples for shipping. Each box weighed 45 pounds. What was the weight of all the boxes?

$n = 360 \times 45$

$n = 16{,}200$

The boxes weighed 16,200 pounds.

13. 573 scouts who attended the Jamboree slept in tents which had 4 beds. How many tents would 573 scouts need?

$573 = (q \times 4) + r$

$573 = (143 \times 4) + 1$

They would need 144 tents.

14. 630 dancers attended the Folk Dance Festival. Into how many groups of 8 could they be divided for square dancing?

$630 = (q \times 8) + r$

$630 = (78 \times 8) + 6$

There could be 78 groups dancing.
6 people could not dance.

Exercise Set 24

Find the numbers q and r must represent to make each sentence true.

1. $994 = (q \times 8) + r$
$994 = (124 \times 8) + 2$

2. $889 = (q \times 7) + r$
$889 = (127 \times 7) + 0$

3. $290 = (q \times 9) + r$
$290 = (32 \times 9) + 2$

4. $493 = (q \times 5) + r$
$493 = (98 \times 5) + 3$

5. $389 = (q \times 4) + r$
$389 = (97 \times 4) + 1$

6. $534 = (q \times 5) + r$
$534 = (106 \times 5) + 4$

7. $954 = (q \times 4) + r$
$954 = (238 \times 4) + 2$

8. $588 = (q \times 6) + r$
$588 = (98 \times 6) + 0$

9. $6769 = (q \times 9) + r$
$6769 = (752 \times 9) + 1$

10. $3626 = (q \times 4) + r$
$3626 = (906 \times 4) + 2$

11. $290 = (q \times 9) + r$
$290 = (32 \times 9) + 2$

12. $5308 = (q \times 7) + r$
$5308 = (758 \wedge 7) + 2$

13. $7449 = (q \times 8) + r$
$7449 = (931 \times 8) + 1$

14. $3636 = (q \times 8) + r$
$3636 = (454 \times 8) + 4$

15. $2390 = (q \times 6) + r$
$2390 = (398 \times 6) + 2$

16. $1235 = (q \times 5) + r$
$1235 = (247 \times 5) + 0$

17. $2770 = (q \times 3) + r$
$2770 = (923 \times 3) + 1$

18. $477 = (q \times 9) + r$
$477 = (53 \times 9) + 0$

19. $6792 = (q \times 7) + r$
$6792 = (970 \times 7) + 2$

20. $493 = (q \times 3) + r$
$493 = (164 \times 3) + 1$

P398

Practice Exercises

I. Place the parentheses correctly to make these true mathematical sentences.

Example: 24 + 6 - 5 = 25 , (24 + 6) - 5 = 25

a) 6 × 9 + 4 = 58 [(6 x 9)+4 = 58]

b) 27 + 13 ÷ 4 = 10 [(27+13)÷4 =10]

c) 9 × 6 + 4 = 90 [9 x(6 + 4) = 90]

d) 7 × 8 + 8 = 112 [7 x(8 + 8) = 112]

e) 7 + 63 ÷ 9 = 14 [7+(63 ÷ 9)=14]

f) 5 × 40 + 8 = 208 [(5 x 40)+ 8 = 208]

g) 7 × 9 - 4 = 35 [7 x(9 - 4) = 35]

h) 35 - 7 ÷ 4 = 7 [(35 - 7)÷ 4 = 7]

i) 43 + 7 ÷ 5 = 10 [(43 + 7)÷5 =10]

j) 54 ÷ 9 + 6 = 12 [(54 ÷ 9)+6 =12]

II. Write the number that n represents

a) n ÷ 4 = 276 (n = 1104)

b) 693 - n = 445 (n = 248)

c) 224 = n × 7 (n = 32)

d) 859 = 384 + n (n = 475)

e) n = 8 × 317 (n = 2,536)

f) 392 + n = 1748 (n = 1,356)

g) 798 - n = 344 (n = 454)

h) 511 ÷ 7 = n (n = 73)

i) 786 + n = 974 (n = 188)

j) 457 + 1066 + 5461 = n (n = 6,984)

III. Add:

25	496	589	32
38	447	9	200
46	582	899	8934
59	785	8938	32
(168)	697	275	3709
	(3,007)	(10,710)	(12,907)

Subtract:

7010	8300	610	9001
6258	7519	352	3729
(752)	(781)	(258)	(5,272)

Multiply:

358	868	69	98
7	8	38	89
(2,506)	(6,944)	(2,622)	(8,722)

Divide

(250)	(201 r1)	(968 r5)	(576)
9)2250	7)1408	8)7749	6)3456

IV. In the chart below write a mathematical sentence then solve it.

Number Pair	Operation	Sentence	Result
25, 40	addition	25 + 40 = a	a = 65
a) 34, 26	multiplication	34 × 26 = n	n = 884
b) 917, 49	subtraction	917 - 49 = n	n = 868
c) 972, 6	division	972 ÷ 6 = n	n = 162
d) 845, 766	addition	845 + 766 = n	n = 1,611
e) 896, 47	multiplication	896 × 47 = n	n = 42,112
f) 3442, 2461	subtraction	3442 - 2461 = n	n = 981
g) 828, 9	division	828 ÷ 9 = n	n = 92
h) 9, 8289	multiplication	9 × 8289 = n	n = 74,601
i) 23334, 6666	addition	23334 + 6666 = n	n = 30,000
j) 768, 8	division	768 ÷ 8 = n	n = 96

V. By regrouping, find the unknown addend.

Example: 462 = 400 + 60 + 2 = 400 + 50 + 12
-157 = 100 + 50 + 7 = 100 + 50 + 7
300 + 0 + 5 = 305

a) 386 300+80+6 = 300+70+16 c) 393 300+90+3 = 300+80+13
-219 200+10+9 = 200+10+9 216 200+10+6 = 200+10+6
100+60+7 = 167 100+70+7 = 177

b) 633 600+30+3 = 500+130+3 d) 761 700+60+1 = 700+50+11
563 500+60+3 = 500+60+3 257 200+50+7 = 200+50+7
70+0 = 70 500+0+4 = 504

VI. Solve the following:

a) 85 × 27 = n (n = 2,295) f) 126 ÷ 3 = n (n = 42)
b) n ÷ 5 = 405 (n = 2,025) g) 600 - n = 568 (n = 32)
c) 9 × 847 = n (n = 7,623) h) 876 + 889 = n (n = 1,765)
d) 352 + n = 900 (n = 548) i) 726 ÷ 8 = n (n = 90 r 6)
e) 27 + 5 + 8 = n (n = 40) j) 9000 - 3402 = n (n = 5,598)

VII. Solve:

a) n + 9 = 97 (n = 88) f) 6 × 7008 = n (n = 42,048)
b) 89 + 95 + 96 = n (n = 280) g) 108 ÷ 5 = n (n = 21 r 3)
c) 10 × 85 = n (n = 850) h) 65 + 54 + 51 + 70 + 33 = n (n = 273)
d) 671 ÷ 9 = n (n = 74 r 5) i) n ÷ 7 = 96 (n = 672)
e) 6040 - n = 2159 (n = 3,881) j) 422 ÷ 6 = n (n = 70 r 2)

VIII. Solve the following:

a) 393 ÷ 8 = n (n = 49 r 1) f) 680 + 807 + 739 = n (n = 2,226)
b) 67 × 36 = n (n = 2,412) g) n + 279 = 871 (n = 592)
c) 64 + 48 + 9 + 85 = n (n = 206) h) 542 - 498 = n (n = 44)
d) 29 + n = 86 (n = 57) i) 547 ÷ 9 = n (n = 60 r 7)
e) 8 × 1321 = n (n = 10,568) j) n ÷ 5 = 5030 (n = 25,150)

IX. Solve:

a) 63 × 80 = n (n = 5,040)

b) 40 + 23 + 16 = n (n = 79)

c) n ÷ 4 = 49 (n = 196)

d) 97 + n = 2005 (n = 1,908)

e) 57 + 30 + 91 = n (n = 178)

f) 278 ÷ 7 = n (n = 39 r 5)

g) 19 × 69 = n (n = 1,311)

h) 357 + 249 + 610 + 8 = n (n = 1,224)

i) 338 ÷ 5 = n (n = 67 r 3)

j) 201 ÷ 4 = n (n = 50 r 1)

Braintwisters

1. Here is an imaginary operation called "cut". The symbol for cut is ∧ . Try to find the meaning of ∧ from these examples.

a) 8 ∧ 4 = 1　　c) 10 ∧ 5 = 1　　e) 2 ∧ 2 = 0

b) 9 ∧ 3 = 2　　d) 20 ∧ 4 = 4　　f) 3 ∧ 1 = 2

(Divide the first number by the second number and subtract one from the answer.)

What is n in each of the following?

g) 35 ∧ 7 = n (n = 4)　　i) 63 ∧ 7 = n (n = 8)　　k) 36 ∧ 6 = n (n = 5)

h) 32 ∧ 8 = n (n = 3)　　j) 56 ∧ 8 = n (n = 6)　　l) 49 ∧ 7 = n (n = 6)

2. Another imaginary operation is called "ler". The symbol for ler is ∫ . Try to find the meaning of ∫ from these examples.

a) 6 ∫ 2 = 3　　c) 10 ∫ 2 = 7　　e) 5 ∫ 1 = 3

b) 12 ∫ 3 = 8　　d) 8 ∫ 7 = 0　　f) 7 ∫ 1 = 5

(Subtract one more than the second number from the first number.)

What is n in each of the following?

g) 25 ∫ 22 = n (n = 2)　　i) 152 ∫ 151 = n (n = 0)　　k) 13 ∫ 6 = n (n = 6)

h) 17 ∫ 5 = n (n = 11)　　j) 72 ∫ 1 = n (n = 70)　　l) 27 ∫ 7 = n (n = 19)

Review

SET I

Part A

1. Use $=$, $>$, or $<$ to make these true statements.

Example: if $n + 2 = 7$, then $n\ (<)\ 7$

a) If $27 \div n = 9$, then $n\ (=)\ 3$

b) If $n + 12 = 17$, then $n\ (<)\ 17$

c) If $n \times 15 = 45$, then $n\ (>)\ 1$

d) If $50 - n = 50$, then $n\ (=)\ 0$

e) If $128 \div n = 32$, then $n\ (=)\ 4$

f) If $n \times 33 = 132$, then $n\ (<)\ 33$

g) If $n \div 7 = 4$, then $n\ (>)\ 4$

h) If $1407 + n = 2989$, then $n\ (>)\ 1407$

i) If $143 = (2 \times 71) + n$, then $n\ (<)\ 2$

j) If $n - 6357 = 653$, then $n\ (>)\ 6357$

2. Write a mathematical sentence to "undo" the following.

Example: $7 + 2 = n$, $(7 + 2) - 2 = n$

a) $31 + 4 = n$ $((31+4)-4=n)$ f) $64 \div 8 = n$ $((64\div8)\times8=n)$

b) $12 \times 6 = n$ $((12\times6)\div6=n)$ g) $125 - 25 = n$ $((125-25)+25=n)$

c) $15 \div 3 = n$ $((15\div3)\times3=n)$ h) $3 \times 3 = n$ $((3\times3)\div3=n)$

d) $423 + 172 = n$ $((423+172)-172=$ i) $427 \div 7 = n$ $((427\div7)\times7=n)$

e) $72 - 13 = n$ $((72-13)+13=n)$ j) $3592 - 1782 = n$
$((3592-1782)+1782=n)$

3. For each multiplication fact write two division facts.

Example: $2 \times 6 = 12$, $12 \div 6 = 2$, $12 \div 2 = 6$

a) $6 \times 7 = 42$ $\begin{pmatrix}42\div6=7\\42\div7=6\end{pmatrix}$ f) $8 \times 6 = 48$ $\begin{pmatrix}48\div8=6\\48\div6=8\end{pmatrix}$

b) $7 \times 8 = 56$ $\begin{pmatrix}56\div7=8\\56\div8=7\end{pmatrix}$ g) $7 \times 4 = 28$ $\begin{pmatrix}28\div7=4\\28\div4=7\end{pmatrix}$

c) $8 \times 9 = 72$ $\begin{pmatrix}72\div8=9\\72\div9=8\end{pmatrix}$ h) $9 \times 5 = 45$ $\begin{pmatrix}45\div9=5\\45\div5=9\end{pmatrix}$

d) $3 \times 8 = 24$ $\begin{pmatrix}24\div3=8\\24\div8=3\end{pmatrix}$ i) $4 \times 8 = 32$ $\begin{pmatrix}32\div4=8\\32\div8=4\end{pmatrix}$

e) $6 \times 9 = 54$ $\begin{pmatrix}54\div9=6\\54\div6=9\end{pmatrix}$ j) $9 \times 7 = 63$ $\begin{pmatrix}63\div9=7\\63\div7=9\end{pmatrix}$

4. Write the correct words or numerals to complete this chart.

Number Pair	Result	Operation	Result	Operation
a) 40, 25	65	Addition	15	Subtraction
b) 72, 8	9	Division	576	Multiplication
c) 96, 8	84	Subtraction	12	Division
d) 84, 23	1932	Multiplication	107	Addition
e) 369, 9	378	Addition	41	Division
f) 80, 12	92	Addition	68	Subtraction
g) 45, 5	225	Multiplication	9	Division
h) 90, 9	81	Subtraction	810	multiplication

5. B ⋃ E = {red, blue, white, green, purple}

B ⋂ E = { }

E = {green, purple}

What operation could you use to find the number of members in Set B? subtraction Name the members of Set B. (red, blue, white)

6. A ⋃ G = {2 , 4 , 5 , 6 , 3 , 7}

A ⋂ G = {5 , 6 , 7}

G = {5 , 6 , 7}

Could you use subtraction to find the number of members in Set A? no Name the members of Set A. (2, 3, 4, 5, 6, 7.)

7. C = {2 , 4 , 6 , 8}

O = {1 , 3 , 5 , 7 , 9}

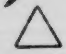
Name the members of the Set C ⋃ O. (1, 2, 3, 4, 5, 6, 7, 8, 9.)

What operation could you use to find the number of members in C ⋃ O ? (addition.)

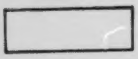
8. Draw a polygon that is the union of

a) 2 line segments not possible

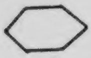
b) 3 line segments



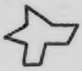
c) 4 line segments



d) 6 line segments



e) 10 line segments



9. How many vertices has each polygon in Problem 8 ?
(a. none b. three c. four d. six e. ten)

10. Find the number that n represents in each of these.

Example a is done for you.

a) $53 + 22 + n = 89$, $53 + 22 = 75$, $89 - 75 = 14$, $n = 14$

b) $24 + 30 + n = 79$ $(n = 25)$

c) $43 + n + 25 = 87$ $(n = 19)$

d) $n + 9 + 30 + 27 = 152$ $(n = 86)$

e) $798 + 9 + n = 1504$ $(n = 697)$

f) $59 + 497 + n + 7 = 1069$ $(n = 506)$

g) $34 + n + 11 = 68$ $(n = 23)$

h) $275 + 596 + n = 1716$ $(n = 845)$

i) $16 + n + 66 = 96$ $(n = 14)$

j) $n + 669 + 352 = 1021$ $(n = 0)$

k) $88 + 7 + n = 174$ $(n = 79)$

11. Match each of the Ideas in Column I with a Model from Column II

Column I
Idea

1) Point (e)
2) Line segment (b)
3) Line (i)
4) Ray (h)
5) Plane (g)
6) Simple Closed Curve (a)
7) Polygon (f)
8) Circle (c)
9) Plane region (d)

Column II
Model

a) My path when I walk all the way around the block and return to my starting point.
b) A stretched piece of string
c) The rim of a drinking glass
d) A football field
e) The tip of a compass
f) The edges of a piece of floor tile
g) The surface of a calm lake whose shores cannot be seen
h) The light from a distant star
i) A straight narrow road with no ends in sight

Part B

Write a mathematical sentence (or two sentences if necessary) for each problem and solve. Write an answer sentence.

1. The Polk Street bus makes three round trips every hour. How many minutes should one round trip take? ($60 \div 3 = n$
$3 \times n = 60$ $n = 20$
One round trip will take 20 minutes)

2. The school cafeteria charges 25 cents for lunch. How much money will a student need for lunches all week? ($.25 \times 5 = l$
$l = 1.25$
a student will need $1.25 for a week's lunches.)

3. Eddie bought 6 tennis balls for $3.18. How much did one ball cost? ($3.18 \div 6 = n$ or $6 \times n = 3.18$ $n = .53$ Eddie paid #.53 for one ball.)

4. A dairy cow requires three acres of grazing land. How much land is needed for 175 dairy cows? ($175 \times 3 = l$ $l = 525$ 525 acres of land are needed.)

5. Mary's baby sister drinks 8 ounces of milk six times a day. How much milk will the baby drink in one week? ($6 \times 8 \times 7 = n$ or $6 \times 8 = p$ $p \times 7 = n$ $n = 336$ Mary's baby drinks 336 ounces of milk in one week.)

6. The class in Room 15 invited their parents to a puppet show. There were only forty-five chairs in the room and 72 parents came. How many parents had to stand? ($72 - 45 = n$ or $45 + n = 72$ $n = 27$ 27 parents had to stand.)

7. One sccut troup delivers 364 hand bills, another troop has 37 less to deliver. How many hand bills do both troops deliver? ($364 - 37 = s$ $s = 327$ $327 + 364 = t$ or $(364 - 37) + 364 = t$ $t = 691$. Both troops deliver 691 hand bills.)

Group Activities

Multiplication Quiz

Child (leader or quiz master) stands in front of the class and says, "I am thinking of two factors whose product is 42." Then he calls on class members.

Child in class group called on asks, "Are you thinking of 6 and 7 ?"

The leader replies yes or no as the case may be. A record is kept on the board of all combinations of numbers correctly given for review later.

If the leader passes a wrong combination he must sit down and a new leader is chosen.

Review

SET II

Part A.

1. Using the symbols $>$, $<$, or $=$ complete these to make true sentences.

a) $747 \underline{(<)} 319 \times 3$

b) $83 \times 7 \underline{(<)} 73 \times 8$

c) $576 \div 9 \underline{(=)} 32 \times 2$

d) $914 - 326 \underline{(>)} 22 \times 25$

e) $34 \times 19 \underline{(=)} 799 - 153$

f) $343 \div 7 \underline{(<)} 49 \times 6$

g) $3148 \underline{(<)} 232 \times 14$

h) $(7 \times 8) \times 2 \underline{(=)} (12 \times 9) + 4$

i) $25 \times 25 \underline{(>)} 30 \times 20$

j) $(40 + 4) \times 4 \underline{(=)} 4 \times 44$

2. Tell whether each of the following is true or false.

a) $6 + 3 = 3 + 6$ (T)

b) $12 - 8 \neq 2 + 2$ (F)

c) $36 + 7 < 35 + 8$ (F)

d) $16 + 12 + 9 = 52 - 15$ (T)

e) $3 \times a$ is always $> 3 \times 2$ (F)

f) $(16 \div 2) \times 2 = 16 \div (2 \times 2)$ (F)

g) $7 \times 6 < 156 - 112$ (T)

h) $29 - 8 \neq 4 \times 7$ (T)

i) $4 \times 6 > 2 \times 11 \times 1$ (T)

j) $6 \times 5 \times 2 \neq 30 + 30$ (F)

3. Tell what operation is used and find r.

Example: $7 \times r = 42$, division , $r = 6$

a) $23 = 14 + r$ (*subtraction r:9*)

b) $r = 5 \times 9$ (*multiplication r:45*)

c) $27 + 14 = r$ (*addition r:41*)

d) $16 + r = 34$ (*subtraction r:18*)

e) $r - 23 = 46$ (*addition r:69*)

f) $24 \times r = 120$ (*division r:5*)

g) $56 \div r = 8$ (*division r:7*)

h) $r = 42 - 16$ (*subtraction r:26*)

4. Write each division sentence as a multiplication sentence. Find the number n represents.

Example: $64 \div 2 = n$, $2 \times n = 64$, $n = 32$

a) $832 \div 4 = n$ ($4 \times n = 832$ $n = 208$)
b) $273 \div 3 = n$ ($3 \times n = 273$ $n = 91$)
c) $568 \div 8 = n$ ($8 \times n = 568$ $n = 71$)
d) $4207 \div 7 = n$ ($7 \times n = 4207$ $n = 601$)
e) $355 \div 5 = n$ ($5 \times n = 355$ $n = 71$)
f) $602 \div 7 = n$ ($7 \times n = 602$ $n = 86$)
g) $664 \div 8 = n$ ($8 \times n = 664$ $n = 83$)
h) $111 \div 3 = n$ ($3 \times n = 111$ $n = 37$)

5. Complete these to make them true sentences using words from this set of words: division, operation, multiplication, addends, subtraction, factor, addition, product.

a) We operate on two factors and get a (product)
b) The operation of (addition) undoes subtraction.
c) We operate on two (addends) and get a sum.
d) The operation of subtraction undoes (addition)
e) To find an unknown addend we use (subtraction)
f) We use division to find an unknown (factor).
g) The operation of (addition) produces a sum.
h) An (operation) on numbers is a way of thinking about two numbers and getting one and only one number.
i) A product is the result of the operation of (multiplication)

6. Complete these to make them true sentences. Find the product.

Example a is done for you.

a) 5 × 14 = (5 × 10) + (5 × 4) = 70

b) 6 × 18 = (6 ×(10)) + (6 × 8) (=108)

c) 9 × 32 = (9 ×(30)) + ((9)× 2) (=288)

d) 7 × 25 = (7 ×(5)) + (7 × 20) (=175)

e) 5 × 82 = ((5)× 80) + (5 ×(2)) (= 410)

f) 25 × 6 = ((20)× 6) + (5 ×(6)) (= 150)

g) 100 × 21 = ((100)× 20) + ((100)× 1) (= 2,100)

h) 32 × 4 = (16 × 4) + ((16)× 4) (= 128)

i) 1000 × 13 = (1000 ×(10)) + ((1000)× 3) (= 13000)

7. Write each of the following using symbols.

Example: The number 8 increased by y , 8 + y

a) The sum of y and 6 (y+6)

b) The number y added to 6 (6+y)

c) The number y increased by six (y+6)

d) Six more than the number y (y+6)

Find the number represented by each of the above if y = 7.
(13)

8. Write each addition sentence as a subtraction sentence.

Find what number n represents.

a) 40 + n = 68 (68-40=n n=28)

b) 36 + n = 39 (39-36=n n=3)

c) n + 54 = 90 (90-54=n n36)

d) 102 + n = 256 (256-102=n n=154)

e) n + 69 = 534 (534-69=n n=465)

f) 452 + n = 931 (931-452=n n=479)

g) 384 + n = 731 (731-384=n n=347)

h) 465 + n = 534 (534-465=n n=69)

9. Match each word in Column I with a picture or a meaning in Column II

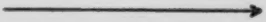
Column I

1) vertex (j)
2) triangle (h)
3) intersection (c)
4) radius (f)
5) quadrilateral (i)
6) plane region (a, e)
7) circle (f)
8) triangular region (e)
9) ray (b)

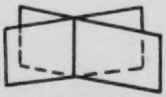
Column II

a) the union of a simple closed curve and its interior

b)



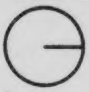
c)



d) the study of space and location

e) the set of points that is the triangle and its interior

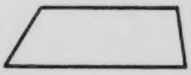
f)



g) the short way to name a line

h) the polygon that is the union of three line segments

i)



j) the common endpoint of two rays that are not on the same line

Part B

Write a mathematical sentence (or two sentences if necessary) for each problem and solve. Write an answer sentence.

1. Coach Lang paid 85 cents each for school softballs. How much does he pay for two dozen softballs? ($2 \times 12 \times 85 = C$ or $85 \times 24 = C$
$C = 20.40$ Coach Lang pays $20.40 for the softballs.)

2. How many ice cream cups can be bought for 90 cents if each cup costs 6 cents? ($90 \div 6 = n$ or $6 \times n = 90$ $n = 15$
15 ice cream cups can be bought for 90 cents.)

3. There were 28 sixth grade girls, 32 fifth grade girls and 30 fourth grade girls at Lincoln School. How many girls were in the three grades?

($28 + 32 + 30 = n$ $n = 90$ There were 90 girls in the three grades.)

4. For his model plane collection, Mark pays $1.29 for one model, $2.25 for another and $1.46 for another. What is the total cost of the models?

($1.29 + 2.25 + 1.46 = c$ $c = 5.00$ Mark pays \$5.00 for the models.)

5. In the problem above, Mark had saved $3.29 and borrowed the remainder from his father. How much did he borrow?

($5.00 - 3.29 = n$ $n = 1.71$ Mark borrowed \$1.71)

6. Barbara can swim 120 yards in 5 minutes. How far can she swim in 20 minutes?

($(120 \div 5) \times 20 = n$ or $120 \div 5 = p$ $p \times 20 = n$ or $(20 \div 5) \times 120 = n$ or $20 \div 5 = t$ $t \times 120 = n$ $n = 480$ Barbara Can swim 480 yards in 20 minutes.)

7. A sign in the bakery reads; cookies - 30 cents a dozen, donuts - 60 cents a dozen, chocolate cakes - 80 cents each. How much does it cost for two dozen cookies and a cake?

($(2 \times 30) + 80 = n$ or $2 \times 30 = c$ $c = 60$ $60 + 80 = n$ $n = 140$ It will cost \$1.40 for the Cake and Cookies)

8. In the problem above, find the cost of two dozen cookies, two dozen donuts and a cake.

$(2 \times 30) + (2 \times 60) + 80 = c$ $c = 260$ The cost is \$2.60.

Individual Projects

1. Make up some operations and their symbols. Work at least 8 problems with each of your make-believe operations. Then put some examples on the board to see if your class can discover their meanings.

2. Arithmetic is only one kind of mathematics. There are at least 79 other kinds. Name five or more other kinds of mathematics. Make a chart for your classroom of the kinds you can find.

Review

SET III

Part A

1. In the chart below, tell which property is illustrated by the number sentence at the left. Write the first letter of each word that names the property instead of writing the words. For example write A P A for Associative Property of Addition.

Number Sentence	Property
a) 320 × 7 = (300 × 7) + (20 × 7)	(D.P.M.)
b) 643 × 29 = 29 × 643	(C.P.M.)
c) 287 ÷ 7 = (280 ÷ 7) + (7 ÷ 7)	(D.P.M.)
d) 381 + (546 + 9) = (381 + 546) + 9	(A.P.A.)
e) 250 ÷ 5 = (200 ÷ 5) + (50 ÷ 5)	(D.P.D.)
f) 37 + 504 = 37 + 504	(None)
g) 46 × 6 = (40 × 6) + (6 × 6)	(D.P.M.)
h) (23 × 7) × 18 = 23 × (7 × 18)	(A.P.M.)

2. Fill in the symbol = or ≠ which makes each of the following a true sentence.

Example: 324 + 415 _≠_ 748

a) 46 + 18 (=) 64

b) 303 + 235 (=) 538

c) 456 - 121 (≠) 337

d) 87 + 344 (=) 431

e) 538 - 382 (=) 156

f) 534 - 273 (≠) 271

g) 56 + 19 + 53 (≠) 148

h) 941 - 327 (≠) 624

i) 897 + 638 (=) 1535

j) 1962 - 1549 (≠) 313

3. Place parentheses in these sentences to make them true.

Example: $4 \times 2 - 1 = 4$, $4 \times (2 - 1) = 4$

a) $23 + 2 \times 5 = 125$ $[(23+2) \times 5 = 125]$

b) $14 \div 2 \times 3 = 21$ $[(14 \div 2) \times 3 = 21]$

c) $30 - 7 + 3 \neq 20$ $[(30-7)+3 \neq 20]$

d) $6 \times 2 - 5 = 7$ $[(6 \times 2)-5 = 7]$

e) $5 + 3 \times 5 \neq 20$ $[(5+3) \times 5 \neq 20]$

f) $6 + 2 \times 3 \neq 12$ $[(6+2) \times 3 \neq 12]$

g) $16 \div 2 \times 4 \neq 2$ $[(16 \div 2) \times 4 \neq 2]$

h) $135 \div 5 + 3 = 30$ $[(135 \div 5)+3 = 30]$

i) $232 \times 6 - 5 = 232$ $[232 \times (6-5) = 232]$

j) $123 \times 3 - 3 = 0$ $[123 \times (3-3) = 0]$

4. Write each of these sentences using numerals and the symbols for "less than" and "greater than".

a) Three is less than five $(3 < 5)$

b) Fifty-eight is greater than thirty $(58 > 30)$

c) Eighteen is less than nineteen $(18 < 19)$

d) Four hundred five is greater than five $(405 > 5)$

e) Three tens are greater than twenty $(30 > 20)$

f) One thousand twelve is less than two thousand $(1012 < 2000)$

g) Seventy is greater than sixty-two $(70 > 62)$

h) Nine hundred ten is less than ten hundred $(910 < 1000)$

i) Three hundred thousand is greater than three thousand $(300{,}000 > 3{,}000)$

j) Forty-six is greater than twenty-six. $(46 > 26)$

5. In the following exercises, use what you know about multiplying by 10 and 100 to get the answers.

Example: 4 × 364 = 1,456 , so 40 × 364 = 14,560

a) 80 × 117 = 9,360 , so 800 × 117 = (93,600)

b) 5 × 766 = 3,830 , so 50 × 766 = (38,300)

c) 9 × 36 = 324 , so 900 × 36 = (32,400)

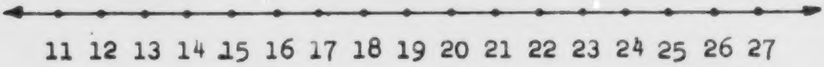
d) 30 × 592 = 17,760 , so 300 × 592 = (177,600)

e) 8 × 125 = 1,000 , so 800 × 125 = (100,000)

f) 3 × 987 = 2,961 , so 30 × 987 = (29,610)

g) 12 × 91 = 1,092 , so 120 × 91 = (10,920)

6.



Using the number line above find how many whole numbers are between these numbers.

a) 13 and 17 (3)

b) 12 and 13 (0)

c) 19 and 11 (7)

d) 19 and 25 (5)

e) 27 and 23 (3)

f) 21 and 20 (0)

g) 15 and 17 (1)

h) 12 and 26 (13)

7. Copy and complete these sentences.

a) A ray has (one) endpoint(s).

b) A triangle is the union of (three) line segment(s).

c) A line has (no) endpoint(s).

d) Space is the set of (all) point(s).

e) A line segment has (two) endpoint(s).

f) A radius is a line segment with (one) endpoint(s) on the circle.

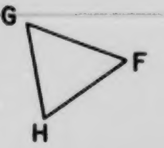
g) A quadrilateral is the union of (four) line segment(s)

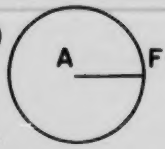
8. Match each word or symbol in Column I with a picture in Column II.

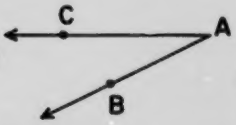
Column I

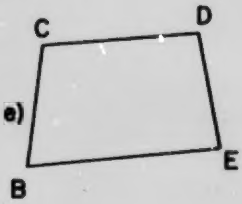
1) $\overrightarrow{AB}$ (c)
2) $\angle CDE$ (e)
3) triangle (a)
4 $\overline{GH}$ (a)
5) $\angle BAC$ (c)
6) triangular region (d)
7) $\overline{AF}$ (b)
8) circle (b)
9) quadrilateral (e)

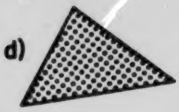
Column II

a)


b)


c)


e)


d)


Part B

Write a mathematical sentence (or two sentences if necessary) for each problem and solve. Write an answer sentence.

1. The Clark family traveled 387 miles in 9 hours. How many miles did they average each hour? (387 ÷ 9 = n n = 43 They averaged 43 miles each hour.)

2. During Public Schools Week, 1,162 people visited Pine Grove School, 1,219 visited Sleepy Hollow School, and 1,094 visited Inland Valley School. How many people visited the three schools? (1162 + 1219 + 1094 = n n = 3475 There were 3,475 people who visited the 3 schools.)

3. Dean and Gail have stamp collections. Dean has 364 stamps. He needs 37 more to have as many stamps as Gail. How many stamps does Gail have? (364 + 37 = n n = 401 Gail has 401 stamps.)

4. The thirty-four children in Room 7 were making bird pictures. The bulletin board would hold 4 dozen pictures. How many children would need to make two pictures? ($4 \times 12 = d$, $d - 34 = n$ $n = 14$ or $(4 \times 12) - 34 = n$ 14 Children will need to make two pictures.)

5. A jet liner averages 449 miles an hour between Los Angeles and St. Louis. The trip takes four hours. How many air miles is it between the two cities? ($449 \times 4 = d$ $d = 1796$ It is 1,796 air miles between the two cities.)

6. How much more do I pay for two shirts that cost $2.15 each than for one shirt that costs $3.29? ($2.15 \times 2 = t$ $t = 4.30$ $4.30 - 3.29 = n$, $n = 1.01$ or $(2 \times 2.15) - 3.29 = n$ I pay $1.01 more.)

7. The price of potatoes is 5 pounds for 29 cents. What is the cost of twenty pounds of potatoes? ($20 \div 5 = p$ $p = 4$ $4 \times 29 = c$ or $(20 \div 5) \times 29 = c$ $c = 116$ The potatoes cost $1.16.)

Group Activity

Tic, Tac, Toe

The objects of the game are speed and accuracy in addition. This is a racing game. Each child draws intersecting line segments as shown. The sum is announced by the teacher. The children put single digit addends in squares so that each row gives the sum.

Example: Sum is 13.

6	5	2
4	3	6
3	6	4
7	4	2

Individual Project

Use only polygons to make an interesting drawing. See how many polygons your classmates can identify.

Chapter 8

RECOGNITION OF COMMON GEOMETRIC FIGURES

PURPOSE OF UNIT

After a short review of triangles and quadrilaterals, the pupil is introduced to several ways of comparing line segments and the idea of congruence. Now he has an opportunity to develop an understanding of isosceles and equilateral triangles and to develop a notion of a right angle. Angles are compared with a constructed model of a right angle to determine if they are "larger than," "smaller than," or "the same as" the model of the right angle. The unit includes a discussion of rectantles and squares.

The work is planned to develop the child's ability to recognize certain common geometric figures and to observe their distinguishing characteristics. The idea of a particular set of points called a simple closed surface is developed. This is done in a way similar to that in which simple closed curve was developed in the unit, Sets of Points.

The simple closed surfaces studied are the prism, cube, pyramid, cylinder, cone, and sphere. Parts of these surfaces are identified with points, lines, and plane figures previously studied.

MATHEMATICAL BACKGROUND

Congruent Line Segments

When we write $4 = 2 + 2$, we mean that 4 and $2 + 2$ are two names for the same number. When we write $\overline{AB} = \overline{CD}$, we mean that $\overline{AB}$ and $\overline{CD}$ are two names for the same line segment; that is, the two segments are the same set of points. If $\overline{AB}$ and $\overline{CD}$ have the same length, but are not the same set of points, we cannot say they are equal and we use the word congruent to describe the relationship.

The work on congruent line segments is needed for a description of isosceles and equilateral triangles in this chapter. A triangle with at least two sides congruent to each other is called an isosceles triangle. A triangle which has three sides congruent to each other is called an equilateral triangle.

Right Angle

A right angle is an angle which has a measure, in degrees, of 90. Since the pupils have not had this concept, they will be taught to recognize a right angle through using models. Many models are available such as the "corner" of a piece of tablet paper, the "corner" of a book, etc. Pupils may construct their own right angle models by folding a sheet of paper twice as is shown in the text.

An angle which is not a right angle may be compared with a right angle. The right angle model can be placed over any model which suggests an angle so that a situation similar to one of the two in Figure I occurs.

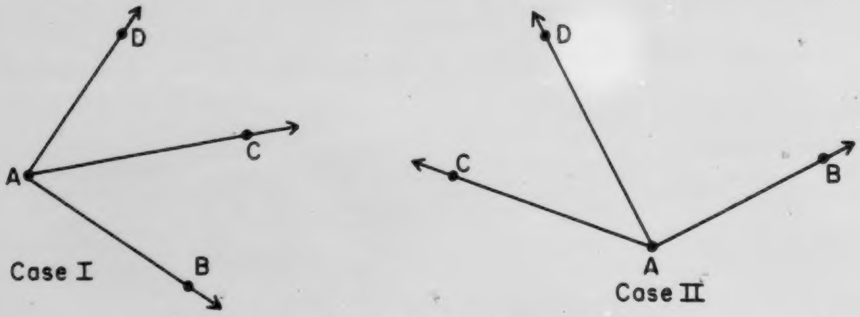


Figure 1

Suppose that the angle to be compared is $\angle BAC$ and $\angle BAD$ is the right angle. In Case I, $\overrightarrow{AC}$ is between $\overrightarrow{AD}$ and $\overrightarrow{AB}$ and we say $\angle BAC$ *is smaller than a right angle*. In Case II, $\overrightarrow{AD}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and we say $\angle BAC$ *is larger than a right angle*.

Although the words *rectangle* and *rectangular* were used earlier, it is in this unit that the meanings are given precisely. A quadrilateral, the angles of which represent right angles, is called a *rectangle*. [One sees at once why we speak of a rectangular prism (see Figure 7 and the paragraph on prisms). Any pair of edges of the rectangular prism with a common endpoint suggests a right angle and each face is a rectangle and its interior.] A rectangle in which all sides are congruent to one another is called a *square*. Some of the surfaces of which paper models will be constructed have faces with edges forming squares. To determine whether or not objects represent rectangles and squares, we use a right angle model and methods for comparing segments.

Simple Closed Surface

The points A, B, and C (which are not on the same line) in Figure 2 determine a plane.

Figure 2

In fact, any three points not on the same line determine a plane, and there is only one plane that passes through these three points.

Now draw line segments connecting points A, B, and C as is shown in Figure 3. The union of $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ is a set of points in a plane. This set of points is called a triangle. Triangles were discussed in the chapter on Sets of Points. The union of triangle ABC and its interior is called a plane region and in this case specifically a triangular region.

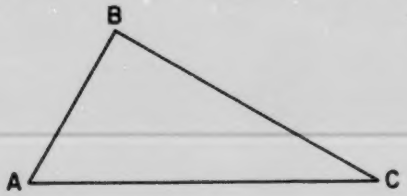


Figure 3

Now let us add to our figure the point D which is not in the plane formed by points A, B, and C. Suppose $\overline{BD}$, $\overline{AD}$, and $\overline{DC}$ are drawn as shown in Figure 4, to form new triangles ABD, ADC, and BDC. Triangle ABD and its interior form a triangular region (shaded in Figure 4), triangle ADC and its interior also form a triangular region, as does triangle BDC and its interior. Together with the triangular region ABC, we now have four triangular regions, no two of which lie in the same plane. The union of these triangular regions is an example of a simple closed surface. In this case, the simple closed surface is called a pyramid. There are many different kinds of pyramids. See page 699 for the background regarding pyramids.

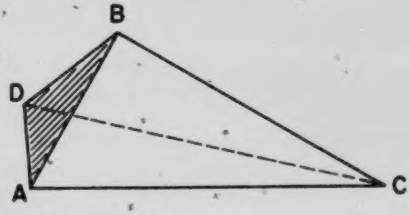


Figure 4

A simple closed surface divides space into three sets of points: the set of points interior to the simple closed surface, the set of points of the simple closed surface, and the set of points exterior to the simple closed surface. One must pass through the simple closed surface to get from an interior point to an exterior point.

In a plane, we called the union of a simple closed curve and points in its interior a plane region. In a similar manner, we will call the union of a simple closed surface and points in its interior a solid region.

This unit is concerned with the recognition of several simple closed surfaces. Also, some additional simple closed curves such as the square and the rectangle are introduced. These are recognized as the union of some of the edges of a simple closed surface.

Prism

The first simple closed surface to be discussed is a prism, which we will define as follows. Consider a polygon. Let a line m not in the plane of the polygon intersect the polygon. Think of all lines such as line p (see Figure 5) which are parallel to line m and intersect the polygon. The union of these lines is the surface shown in in Figure 6a. The surface extends indefinitely up and down. Now let the surface be cut by two parallel planes. The two "cuts" are polygons. The portion between the two parallel planes together with the portion of the planes (you might call them the top and the bottom, or better, the bases) which are interior to the "cuts" form a simple closed surface. It is called a prism. Observe that a prism, as we are using the term, is hollow.

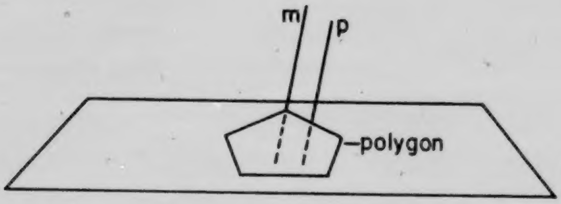


Figure 5

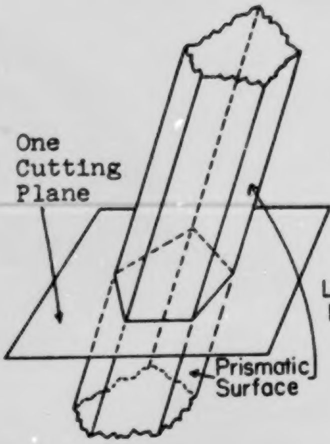

One
Cutting
Plane
Prismatic
Surface

Figure 6a

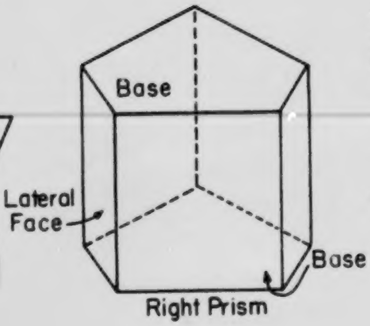

Base
Lateral
Face
Base
Right Prism

Figure 6b

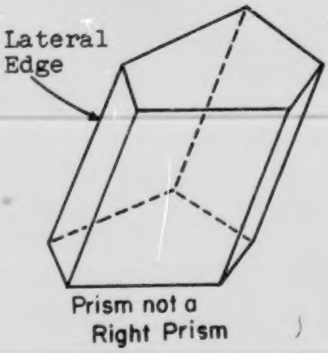

Lateral
Edge
Prism not a
Right Prism

Figure 6c

Each of the plane regions is called a face of the prism. The two faces formed by the parallel planes are called the bases and the other faces are called lateral faces. (The word "lateral" is not used in the pupil text.) The intersection of two adjacent faces of a prism is a line segment, called an edge. The intersection of lateral faces are called lateral edges. Here the use of the word edge is suggested by the meaning of the word as it is commonly used. Each endpoint of an edge is called a vertex. The plural of vertex is vertices.

If the polygon outlining the base is a triangle, the prism is called a triangular prism. A prism is called a quadrangular prism if the polygon is a quadrilateral, and a pentagonal prism if the polygon is a pentagon. The special quadrangular prism in which the quadrilateral is a rectangle is called a rectangular prism. A special rectangular prism in which all edges are congruent to each other is called a cube. The only prisms the pupils are expected to identify in this chapter are the rectangular prism (including the cube) and the triangular prism. See Figure 7.

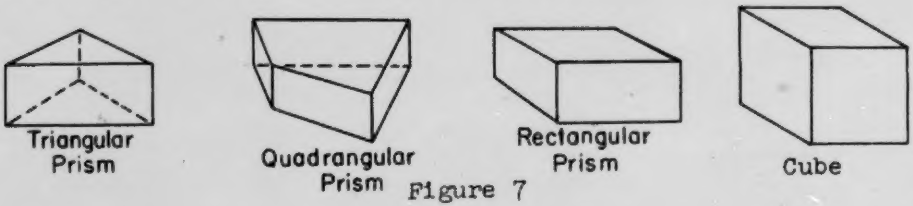

Triangular
Prism
Quadrangular
Prism
Rectangular
Prism
Cube

Figure 7

All of the prisms in Figure 7 are examples of a special type of prism called a right prism in which the lateral edges are perpendicular to the base. All prisms in the pupil text are right prisms and the terminology "right prism" is not introduced. A prism which is not a right prism is shown in Figure 6c.

Pyramid

In defining a pyramid, we will proceed somewhat as we did for the prism. This time think of a point P not in the plane of the polygon and all lines that intersect the polygon and pass through the point P. The point P is called the vertex. The union of these lines is a surface such as is shown in Figure 8a.

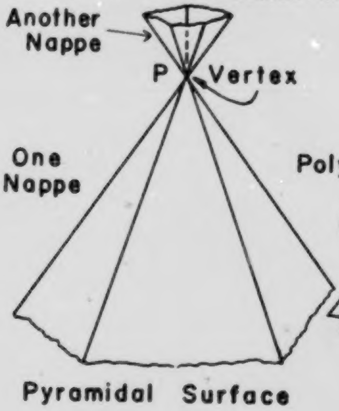


Figure 8a

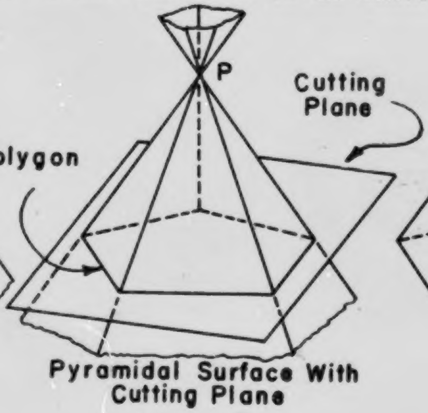


Figure 8b

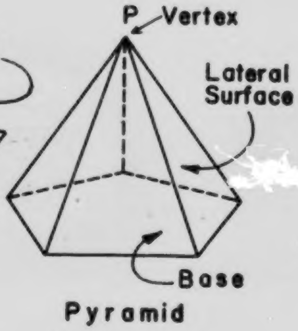
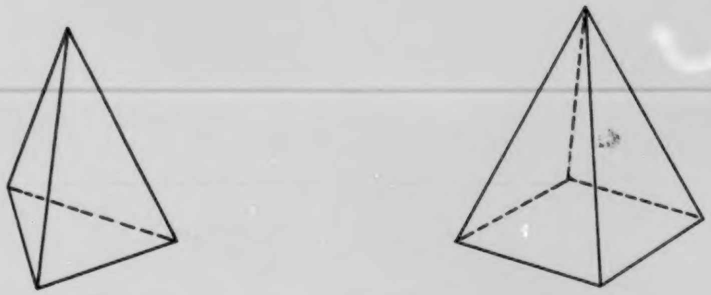


Figure 8c

This surface consists of two parts separated by the vertex. Each of these parts is called a nappe (Figure 8a). Cut one nappe by a plane in such a way that the intersection is a polygon (Figure 8b). This polygon and its interior is called the base. The union of the base, the vertex, and the portion of the nappe "between" the vertex and the base is a simple closed surface called a pyramid (Figure 8c). Observe that a pyramid, as we are using the term, is hollow. The triangular regions are called lateral faces. In the pupil's book, the lateral faces are just called faces.

As with the prism, a pyramid is classified as triangular, quadrangular, pentagonal, etc., according to whether the polygon outlining the base is a triangle, quadrilateral, pentagon, etc. See Figure 9 on the following page.



Triangular Pyramid

(Also called a tetrahedron)

Square Pyramid

Figure 9

Cylinder

A cylinder is defined in a manner similar to the way we defined a prism. In fact, in a very general sense, a prism is just a special case of a cylinder. Does this surprise you? This time, rather than starting with a polygon, let us start with any simple closed curve (it could be a polygon, of course).

We will proceed exactly as we did in defining the prism. Let a line m not in the plane of the simple closed curve intersect the curve. Think of all lines which are parallel to line m and intersect the curve. The union of these lines is a surface. Now let the surface be cut by two parallel planes. The intersection of the surface with either of these two parallel planes is a simple closed curve. Each of these curves, together with its interior, is called a base. The union of the two bases and the portion of the surface between the two bases is a simple closed surface called a cylinder. Figure 10 shows some examples of cylinders.

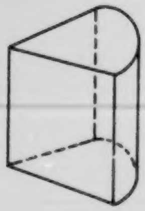


Figure 10a

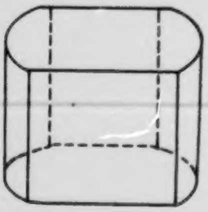
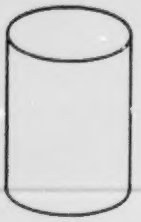
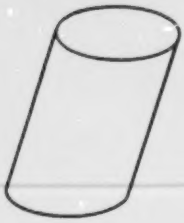


Figure 10b



Right
Circular
Cylinder

Figure 10c



Circular Cylinder
which is not a
right cylinder

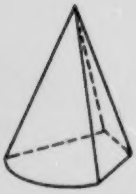
Figure 10d

Figure 10c is a special type of cylinder in which the simple closed curve is a circle and is called a circular cylinder. As with the prism, we will consider only right cylinders where line m is perpendicular to the base.

Cone

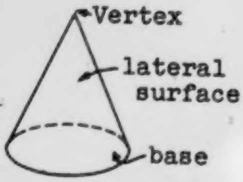
If, in the definition of a pyramid, we replace the word polygon by "simple closed curve" we obtain a cone. Thus, in a very general sense, a pyramid is a special type of cone.

If the simple closed curve is a circle, we get a circular cone. A cone has a base, a lateral surface, and a vertex. The lateral surface is not called a face, since it is not a plane region. See Figure 11b.



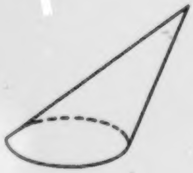
Cone

Figure 11a



Right Circular
Cone

Figure 11b



Circular cone
not a right cone

Figure 11c

Sphere

We have defined a circle as a set of points *in a plane* such that every point of the set is the same distance from a point in a plane called the center. The definition of a *sphere* is the same as the definition of a circle except that the phrase "in a plane" is omitted. For example, a sphere of radius 4 inches consists of all points *in space* 4 inches from a fixed point. The fixed point is called the *center* of the sphere. The center is not a point of the sphere. Any point in space whose distance from the center is less than 4 inches is in the interior of this sphere. Any point in space whose distance from the center is more than 4 inches is in the exterior of this sphere.

The surface of a globe is an example of a sphere. The equator and the lines of longitude represent some of the circles that compose the sphere. Observe that a sphere, as we are using the term, is hollow.

REVIEW OF TRIANGLE AND QUADRILATERAL

Objective: To review understandings of the properties of the triangle and the quadrilateral developed in a previous unit.

Materials needed: Chalkboard, chalk, paper, pencil, straightedge, pegboard, pegs, crayon, string

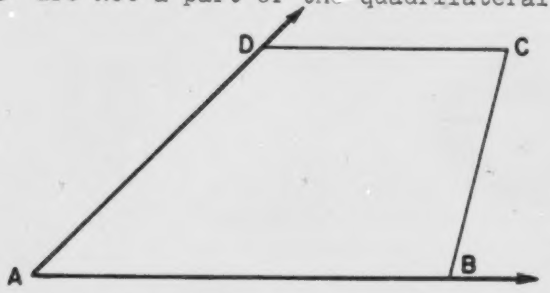
Vocabulary: Sides, vertex, vertices, triangle, quadrilateral

Suggested Teaching Procedure:

The teacher may want to use pegs and a pegboard in this review section. Pegs can be placed at selected points of a pegboard. These points may represent the vertices of either triangles or quadrilaterals. Vertices may be joined by string or rubber bands. The idea of sides and vertices can be easily developed. Pupils might then open their books and quickly work through, with the teacher, the Thinking Together exercises. They will do the On Your Own exercises independently.

In connection with the last three paragraphs on page 418 of the pupil text, the following discussion might be helpful. Draw on the board a figure similar to the one below and say,

Let us examine one of these angles, say $\angle DAB$. The sides $\overline{AD}$ and $\overline{AB}$ are segments and not the entire rays of the angle. $\angle DAB$ is shown in heavy lines below. This shows that the angles of the polygon ABCD are not a part of the quadrilateral.



It may be necessary to show, in a similar way, that $\angle ABC$, $\angle BCD$, and $\angle CDA$ are not a part of the quadrilateral.

A discussion to demonstrate this same idea was done with a triangle in Sets of Points, so children might not have any difficulty with this idea.

P417

Chapter 8

RECOGNITION OF COMMON GEOMETRIC FIGURES

REVIEW OF TRIANGLE AND QUADRILATERAL

Thinking Together

1) a) What name is given to a polygon which is the union of three line segments? (triangle)

The three line segments are called the sides of the triangle.

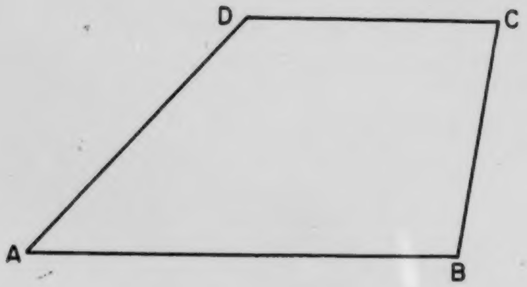
b) What is the common endpoint of any two sides of a triangle called? (a vertex)

c) What are the endpoints of the line segments of a triangle called? (vertices)

d) How many sides and vertices does a triangle have? (3 sides and 3 vertices)

2) What is a polygon which is the union of four line segments called? (quadrilateral)

We call the four line segments of the quadrilateral the sides of the quadrilateral. Below is a drawing of a quadrilateral with sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$.



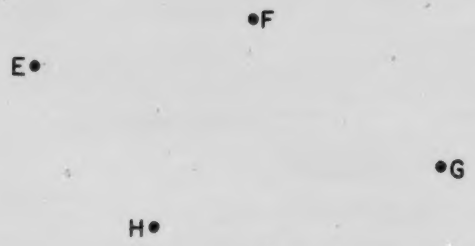
Let us recall that quad suggests four. In the figure shown above four angles are formed. They are $\angle DAB$, $\angle ABC$, $\angle BCD$, and $\angle CDA$. There are some points of these angles that are not points of the quadrilateral because angles are made up of rays.

The vertex of one of the angles is called a vertex of the quadrilateral.

The vertices of these four angles are called the vertices of the quadrilateral.

Exercise Set 1

1. a) Four points are marked below. Trace these points on a sheet of paper and label them as shown.



b) Draw $\overline{EF}$, $\overline{FG}$, $\overline{GH}$, $\overline{HE}$.

2. On the sheet of paper on which you drew the figure for exercise 1 write answers to the following questions:

a) Do these segments form a quadrilateral? (Yes)

b) Name the sides of this quadrilateral. ($\overline{EF}$ or $\overline{FE}$, $\overline{GH}$ or $\overline{HG}$, $\overline{FG}$ or $\overline{GF}$, $\overline{EH}$ or $\overline{HE}$)

c) Name a vertex of the quadrilateral. (Answers will vary. Possibilities are E, F, G, and H.)

d) Name the vertices of this quadrilateral. (E, F, G, H)

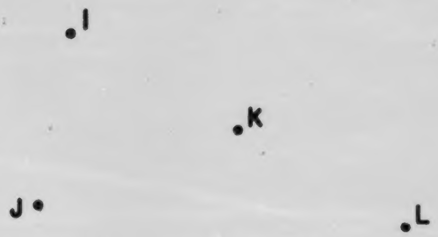
e) Color the interior of the quadrilateral.
(Color only the interior. Do not color the quadrilateral EFGH. This would be a good time to review the concept of region.)

3. Go back to exercise 1. Trace the 4 points again on a sheet of paper and label them as shown.

a) Draw $\overline{EG}$, $\overline{FH}$, $\overline{EH}$, and $\overline{FG}$.

b) Do these segments form a quadrilateral? (No) Why? (They do not even form a polygon.)

4. a) Four points are marked below. Trace these points on a sheet of paper and label them as shown.



b) Draw $\overline{IJ}$, $\overline{JL}$, $\overline{LK}$, and $\overline{KI}$.

5. On the sheet of paper on which you drew the figure for exercise 4, write answers to the following questions.

a) Is your figure a union of four line segments? (yes)

b) Do $\overline{IK}$ and $\overline{KL}$ lie on the same line? (yes, at least they are supposed to.)

c) Is your figure a quadrilateral? (no) Why or why not?
(Because it is also a union of the three line segments $\overline{IJ}$, $\overline{JL}$, and $\overline{LI}$.)

d) Is your figure a polygon? (yes)

e) What is a name for your figure? (triangle)

6. Mark three points (not all on the same line) on your paper. Label them P, Q, and R. Draw $\overline{PQ}$, $\overline{QR}$, and $\overline{RP}$.

a) Is your figure a polygon? (yes)

b) Is your figure the union of three line segments? (yes)

c) What is a name for your figure? (triangle)

d) What is a triangle? (A polygon which is a union of three line segments.)

COMPARING LINE SEGMENTS

Objective: To develop methods of comparing line segments and to develop the notion of congruence.

Materials: String, compass, straightedge, pegboard, pegs, paper, pencil, chalkboard, chalk

Vocabulary: Congruent

Suggested Teaching Procedure:

Review the instructions for using a compass as taught in the lesson on circles in Chapter 5, Sets of Points.

In making comparisons of line segments, a compass, where it can be used, is probably the best device. In other cases, a piece of string may be the only feasible instrument of comparison. Superimposed tracing is a third method.

These three methods all involve the use of models. As you read the pupil's book you will find that the use of models in comparing line segments is carefully described. A tracing and a piece of string obviously are models. It is not so apprarent that we are using a model when we use a compass, because only the endpoints are indicated.

If two models of line segments can be matched endpoint for endpoint with each other, we say they are congruent or we say that one segment is congruent to the other.

To begin the discussion, ask the children to compare by eye alone two segments represented by objects in the room. They should check this comparison by making a comparison using a piece of string. Ask the children if the comparison made by using their eyes is as accurate as the comparison made by using a piece of string. Next show the children two different pieces of chalk and ask them how they can accurately compare the two. A child might answer that he can either use a string or place the two pieces of chalk next to each other to make a comparison. It is possible that a child might suggest the use of a compass to compare the two segments represented by drawings. At this time allow the child time to experiment and to demonstrate his ideas to others. The teacher should guide him by suggestions, directions, or demonstration so that he achieves success in his endeavor.

The Pupil Book is in sufficient detail to be followed.

COMPARING LINE SEGMENTS

Thinking Together

1) Henry says he thinks the line segment represented by the edge of his desk is longer than the line segment represented by the bottom edge of the door. Bill thinks differently. They have only a long piece of string. How can they find out which segment is longer?

2) Bill says, "I can take this long piece of string and hold it at one corner of your desk. Then we can extend the string along the edge to the other corner. Let us hold this string so that it represents the edge of the desk."

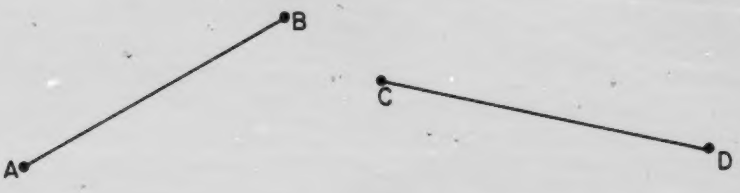
"Henry, you hold your end of the string at one corner of the bottom edge of the door. I place the string along the edge leading to the other corner. Suppose the string does not reach the other corner. Then the segment represented by the bottom edge of the door is longer than that represented by the edge of your desk. If the string goes beyond the other corner, then the edge of the door is shorter than the edge of the desk. If the string matches exactly the bottom edge of the door from corner to corner, the line segments represented are congruent.

Bill says the line segment represented by the edge which runs from the top to the bottom of the door is longer than the line segment connecting the corner of his desk to the teacher's desk. How can Bill decide which is longer?

You can always compare line segments which are represented by objects if you have a piece of string that is long enough.

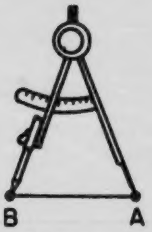
Another way to compare line segments is to use a compass.

If you had only a compass and you wished to decide which of the represented line segments below is the longer, how would you do it?



3) Follow the directions and you will learn how a compass may be used to compare the line segments represented above. On a separate sheet of paper trace the above figure.

a) $\overline{CD}$ is part of $\overrightarrow{CD}$. Extend $\overrightarrow{CD}$ to the edge of the sheet of paper.



b) Set the metal tip of the compass on A and the pencil tip of the compass on B.

c) Without changing the setting of your compass, place the metal tip at C as if you were going to draw the circle with center at C.

d) Draw just enough of the circle to intersect $\overrightarrow{CD}$.

e) Label the intersection T. If T is between C and D, then $\overline{AB}$ is shorter than $\overline{CD}$. If D is between C and T, then $\overline{AB}$ is longer than $\overline{CD}$. If the point of intersection is D, then $\overline{AB}$ is congruent to $\overline{CD}$ (or $\overline{AB}$ and $\overline{CD}$ are congruent segments).

4) a) Compare the segments with a piece of string.

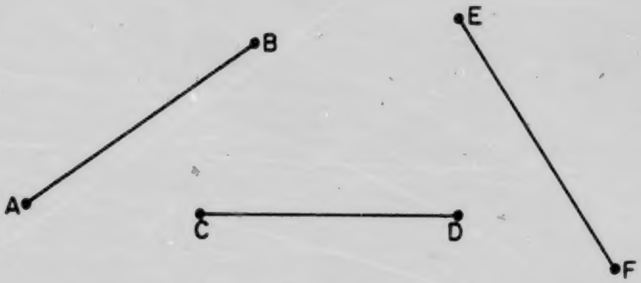
b) Which way of comparing do you think is better here? (Compass)

5) Trace the three line segments. Compare their lengths. Use a compass.

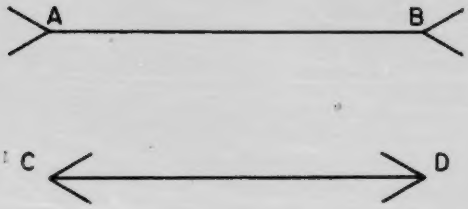
a) Name the shortest line segment. ($\overline{CD}$)

b) Name the next shortest. ($\overline{AB}$)

c) Name the longest. ($\overline{EF}$)



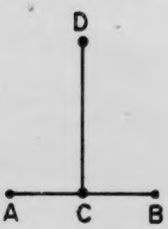
6) Look at $\overline{AB}$ and $\overline{CD}$ below. Which appears to be longer? Use your compass to compare $\overline{AB}$ and $\overline{CD}$. Which is longer? (Neither, they are congruent.)



7) Compare $\overline{AB}$ and $\overline{CD}$.

Which is longer?

(Neither, they are congruent.)



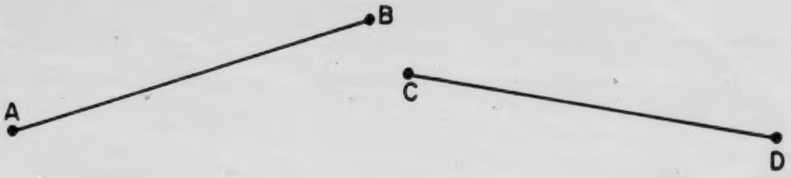
Exercises 6 and 7 shows us that sometimes we must use an instrument to compare segments. We cannot trust our eyes alone.

8) The line segments represented below are congruent.

a) Show that they are congruent by using your compass.

b) Place a sheet of thin paper over $\overline{AB}$ and trace it.

c) Move the tracing so that the dot marked A covers point C. Can you make the dot marked B cover D? The tracing of $\overline{AB}$ matches the drawing of $\overline{CD}$ exactly, endpoint for endpoint.



Of course, we have not actually moved $\overline{AB}$. We have moved a drawing of it.

9) If someone asked you to compare segments, which method would you use? (Have a general discussion comparing the three methods. Each method is used in different circumstances. When it can be used, the compass method is more accurate.)

ISOSCELES AND EQUILATERAL TRIANGLES

Objective: To develop an understanding of properties of isosceles and equilateral triangles.

a) An equilateral triangle has three sides congruent to one another.

b) An isosceles triangle has at least two sides congruent to each other.

Materials: Compass, straightedge, string, paper, pencil, chalk and chalkboard, wire or stick representations of isosceles and equilateral triangles, models (paper) of prisms and pyramids for display.

Vocabulary: Isosceles, equilateral

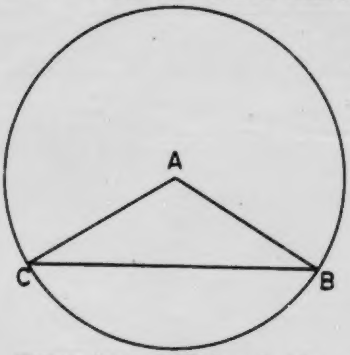
Suggested Teaching Procedure:

Prior to the lesson, the teacher should construct an equilateral triangle and an isosceles triangle out of wire or sticks.

The procedure for constructing isosceles and equilateral triangles on paper with compass and straightedge is given below.

1) Construction of an isosceles triangle.

a) Draw a circle with center at A.



b) Draw two radii of the circle (not on the same line).

c) Label the endpoints of these radii, which are also points, of the circle, B and C.

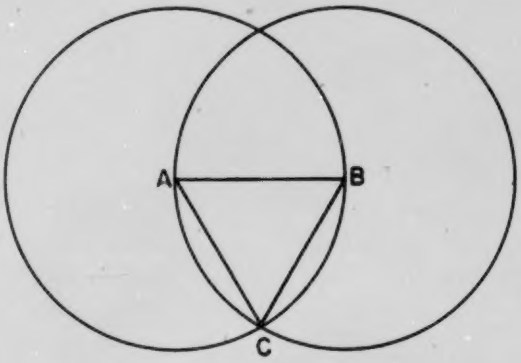
d) Draw $\overline{BC}$.

e) $\triangle$ ACB is an isosceles triangle.

f) $\overline{AC}$ and $\overline{AB}$ are congruent.

2) Construction of an equilateral triangle with a chosen line segment AB as one side.

a) Draw a circle with center at A and radius $\overline{AB}$.



b) Without changing the opening of the compass, draw a circle with center point at B.

c) Label one of the points where the two circles intersect, C.

d) Draw $\overline{AC}$ and $\overline{BC}$.

e) $\triangle ABC$ is an equilateral triangle.

f) $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ are congruent.

The children will find it interesting to make a bulletin board display of geometric forms found in nature. They may bring to class leaves, shells, arrow heads; and pictures of insects, animals, snowflakes, crystals, volcanoes. The children should classify and label these representations of geometric forms.

Prior to Exercise Set 2 you will need to show the children how to construct an isosceles triangle and an equilateral triangle, using only a compass and a straightedge.

ISOSCELES AND EQUILATERAL TRIANGLES

Thinking Together

A triangle which has at least two sides congruent to each other is called an isosceles triangle.

A triangle which has three sides congruent to each other is called an equilateral triangle.

1. Which of the triangles below is an isosceles triangle? (both) Which is equilateral? (Figure 1)

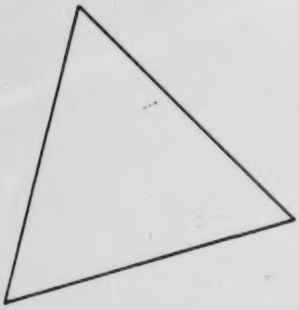


Figure 1

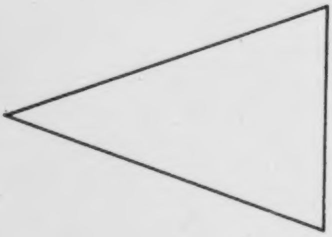
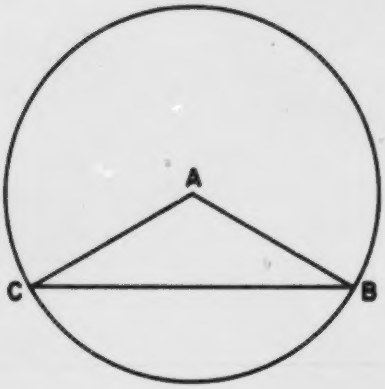


Figure 2

2. Are there any isosceles triangles suggested by the edges of the models in your classroom?
(It will depend upon the models.)
3. Are there any equilateral triangles suggested by the edges of the same models? (It will depend upon the models)
4. Name some things on which you see isosceles triangles represented.
5. Name some things on which you see equilateral triangles represented.

Exercise Set 2

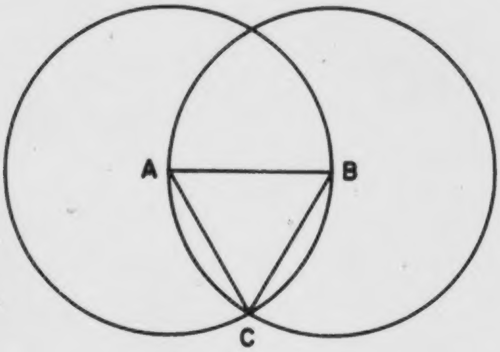
1. a) Draw an isosceles triangle using only a compass, pencil, and a straightedge. This picture may help you follow the directions of your teacher.



b) Is $\overline{AC}$ congruent to $\overline{AB}$? (Yes)

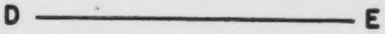
Why? (Because the radii of a circle all have the same length.)

2. a) Draw an equilateral triangle using only a compass, pencil, and a straightedge. This picture may help you follow the directions of your teacher.



b) Is $\overline{AB}$ congruent to $\overline{BC}$ and to $\overline{AC}$? (yes)
Why? (Because the radii of a circle all have the same length.)

3. Draw an isosceles triangle which is not also an equilateral triangle. Make one of its sides congruent to $\overline{DE}$.



4. Draw an equilateral triangle. Make each of its sides congruent to $\overline{DE}$ of exercise 3.

RIGHT ANGLES

Objective:	To develop the concept of a right angle, and then to compare angles with a constructed model of a right angle.
Materials:	Paper, pencil, chalk, chalkboard, models (paper) of prisms and pyramids for display
Vocabulary:	Right angles

Suggested Teaching Procedure:

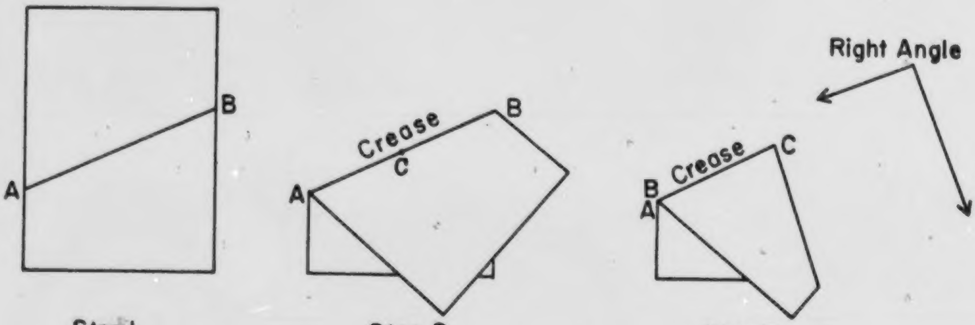
> The presentation in the Pupil Text is in sufficient detail to be followed.

RIGHT ANGLES

Thinking Together

We have learned that an angle is the union of two rays which are not on the same line. The two rays must have a common endpoint. By folding paper we are going to represent an angle which is called a right angle.

Fold a sheet of paper. It is not necessary to have the edges even. The crease represents a line segment. Now fold the paper again so that the edges of the first crease line up exactly. The intersection of the two creases is the vertex of an angle. The creases represent part of the rays of an angle. Show these rays. The angle that is represented is called a right angle.



Step 1 Step 2 Step 3

Does this page itself suggest a model of a right angle? (Yes, a corner)

Name some other models of right angles in your classroom. (Answers will vary.)

We shall use the right angle model to draw a right angle having a chosen ray as one of its rays. Let the ray represented below be the chosen ray. Draw a ray like this on a sheet of paper.

1. a) Label the endpoint of the ray as A.
   b) Place the folded paper model of the right angle so that the vertex is at A and one of the creases lies along the ray.
   c) Trace along the other crease from the vertex.
2. There are two possible right angles which can be represented on the paper if the instructions above are followed. Draw both of these.

We shall use our model to compare angles with a right angle. Suppose we wish to compare ∠BAC, represented in Figure 1, with a right angle. On another sheet of paper copy ∠BAC. Then draw the right angle ∠BAD with $\overrightarrow{AB}$ as one of its rays. Draw it so that D is on the same side of $\overrightarrow{AB}$ as C. Notice that $\overrightarrow{AC}$ falls between $\overrightarrow{AB}$ and $\overrightarrow{AD}$. We shall say, therefore, that ∠BAC is smaller than ∠BAD. So ∠BAC is smaller than a right angle.

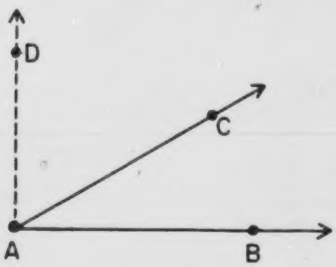


Figure 1

Suppose instead of the picture of the previous figure, we have the picture below. Here $\overrightarrow{AD}$ is between $\overrightarrow{AC}$ and $\overrightarrow{AB}$. So we say that

∠BAC is greater than a right angle.

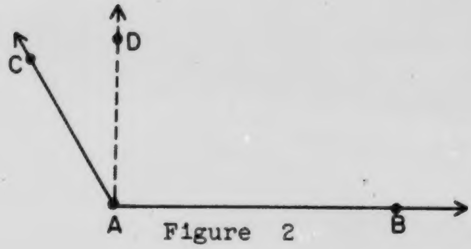


Figure 2

Of course, if C is on $\overrightarrow{AD}$, then ∠BAC is a right angle.

Thinking Together

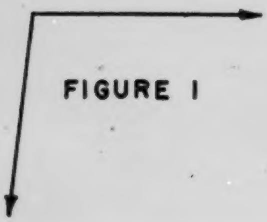


FIGURE 1

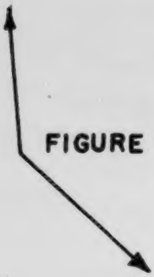


FIGURE 2

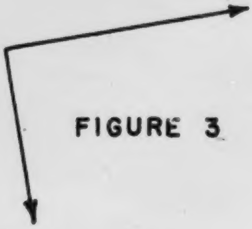


FIGURE 3

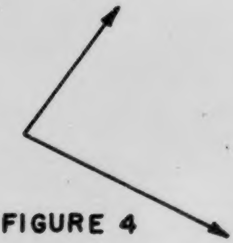


FIGURE 4

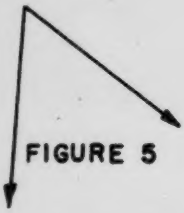


FIGURE 5

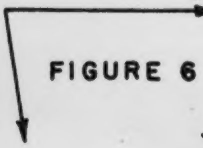


FIGURE 6

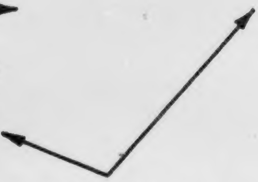


FIGURE 7

1. In this picture seven angles are represented. Look at each angle carefully. Without using a right angle model, name those angles which you think are right angles.
(Answers will vary. Figure 3 is a picture of a right angle.)

2. Without using a right angle model, name those angles which seem to be greater than a right angle. (Ones of Figures 1, 2, 7)

3. Without using a right angle model, name those angles which seem to be smaller than a right angle. (Ones of Figures 4, 5, 6)

4. Now check the figures with a folded paper model of a right angle to see if your answers are correct.

5. Are your answers in Exercise 4 the same as your answers in Exercises 1 - 3? (Answers will vary.)

6. You see how an angle may be compared with a right angle if the angle has been represented by a drawing. Suppose an object suggests an angle. How would you compare this angle with a right angle? Find some object in your classroom which suggests an angle and compare it with your model of a right angle.

7. Are the angles represented by the edges of your desk right angles? (Probably)

8. a) Do the hands of a clock ever suggest right angles? (yes)

   b) Name a time when they suggest an angle less than a right angle. (There are many possible answers such as ten minutes past twelve.)

   c) Name a time when they suggest an angle greater than a right angle. (There are many possible answers such as twenty minutes past twelve.)

9. Sue wants to know if the angle the hands of the clock suggest when the time is 12:10 is larger or smaller than a right angle. She has a tracing of a right angle on thin paper. How might she use this tracing to decide whether or not the angle is larger or smaller than a right angle?

10. Name some other objects which suggest right angles.
(Intersection of some streets, the edges of a post card, the blades of a pair of scissors may be opened to suggest a right angle, a door may be opened so that the opening suggests a right angle.)

RECTANGLES AND SQUARES

Objective: To develop an understanding of properties of rectangles and squares. A quadrilateral which has four right angles is called a rectangle; a rectangle which has all its sides congruent to one another is called a square.

Materials: Paper, pencil, chalk, chalkboard, compass, string, straightedge, models (paper) of pyramids and prisms for display.

Vocabulary: rectangle, square

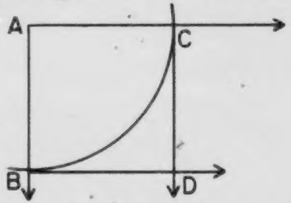
Suggested Teaching Procedure:

The presentation in the Pupil Book is in sufficient detail to be followed.

Answer to Braintwister on Page 435 of Pupil Text.

Draw a square using only the folded paper model of a right angle, a compass, and a pencil.

With a right angle model we can represent a square.



a) Choose a point and label it A.
b) With the right angle model made of folded paper, draw a right angle with vertex A.
c) Draw a circle with center at point A and radius any convenient length.
d) Label as B and C the two points of intersection of the circle and rays of the right angle.
e) Draw a right angle with vertex at B, having $\overrightarrow{BA}$ as a ray, as in the picture.
f) Draw a right angle with vertex at C, having $\overrightarrow{CA}$ as a ray, as in the picture.
g) The right angles with vertices C and B have rays which intersect in point A and another point. Label this point D.
h) Compare the angle $\angle BDC$ with your right angle model. It is a right angle.

The rectangle with vertices A, B, C, D, is a square. Testing with a compass will show that the sides are congruent.

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RECTANGLES AND SQUARES

Thinking Together

If each of the four angles of a quadrilateral is a right angle, we say that the quadrilateral is a rectangle.

1. Can you find edges of your book which represent a rectangle? (yes) Check with your right angle model.

2. Do the edges of this sheet of paper represent a rectangle? (yes) Check with your right angle model.

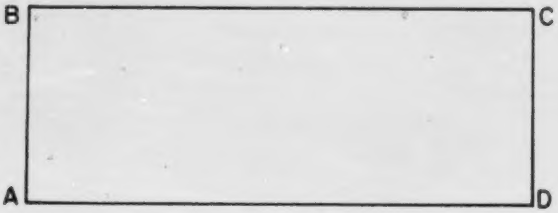
3. Can you find any models in your classroom which represent a rectangle? (yes) Check with your right angle model.

4. Name some objects at your home which suggest rectangles. (Tops of desks, surfaces of a door, some tables, some windows, a sand box, parts of sidewalks, the cover of a book.)

5. How could you draw a rectangle using your right angle model? (Answers will vary.)

Exercise Set 4

1. a) Is the quadrilateral represented below with vertices A, B, C, and D a rectangle? (yes) Use your model of a right angle.



b) Is $\overline{AB}$ congruent to $\overline{BC}$? (no)

c) Is $\overline{AB}$ congruent to $\overline{DC}$? (yes)

d) Is $\overline{BC}$ congruent to $\overline{AD}$? (yes)

e) Is $\overline{BC}$ congruent to $\overline{CD}$? (no)

2. Which of these statements is true?

a) A rectangle has two pairs of congruent sides. (true)

b) All four sides of every rectangle are congruent. (false)

c) A rectangle has four right angles. (true)

3. Make a copy of $\overline{AD}$ of exercise 1. Using $\overline{AD}$ as one side, see if you can draw another rectangle which looks different from rectangle ABCD of exercise 1.

Thinking Together

A rectangle with all its sides congruent to one another is called a square.

1. Below is a representation of a square. Check the angles with your right angle model and the sides with your compass.

   a) Are all the sides congruent to each other. (yes)

   b) Are all the angles right angles? (yes)

2. Name some objects in your classroom which suggest squares. (faces of blocks, faces of asphalt tile)

3. Name some objects in your home which suggest squares. (Answers will vary.)

4. Is every square a rectangle? (yes)

5. a) Is every rectangle a square? (No)

   b) Are some rectangles also squares? (yes)

BRAINTWISTER

Draw a square using only the folded paper model of a right angle, a compass, and a pencil. (See TC page 724 for solution.)

SURFACES

Objective: Recognition of some common surfaces

Materials: Paper, pencil, compass, chalk, chalkboard. Geometric models of a rectangular, square, and triangular prism; circular cylinder; non-circular cylinder; sphere; cone; square, rectangular, and triangular pyramid; square. Objects that have these geometric shapes, such as cereal boxes, cylindrical ice cream cartons, non-circular cylindrical tooth brush containers, drum, drinking glass, funnel, blocks of wood or plastic boxes in the form of a rectangular prism, and plastic, wood, or paper models.

Vocabulary: Interior, square prism, rectangular prism, triangular prism, cylinder, circular cylinder, cone, face, edge, rectangular pyramid, square pyramid, sphere.

Suggested Teaching Procedure:

One way to establish the background for the ideas that will be developed in this lesson: Prior to the lesson (this could mean just before the children go home the preceding day or at the end of the previous day's lesson) take about fifteen minutes to show geometric models and the photographs of a rectangular and triangular prism, cylinder, cone, square pyramid, triangular pyramid, and rectangular pyramid. The children will be able to name some of these geometric forms.

Direct the discussion and supply the correct names which will appear below the photographs in the text. Then ask the children to bring to school the following day objects that resemble these models shown. The geometric forms listed in the materials section should be suggested to the children. (The teacher should have a supply of these models.)

As the lesson is presented, each child should have an object which has the geometric shape being discussed so that he can follow the discussion and indicate the various parts of these objects as they are explained. The terms employed here will be used in a descriptive sense and not in a precise way.

Throughout this section, it must be understood that the models we are using are just representations of sets of points in space.

Prism

A discussion of the prism is given in the mathematical background section of this unit.

Perhaps the most difficult thing for the children to understand is that a prism is just the surface of the figure. That is, a prism is "hollow". It is composed of a finite number of portions of planes. We will consider the points that are "inside" of the prism in the unit on Volume in grade six.

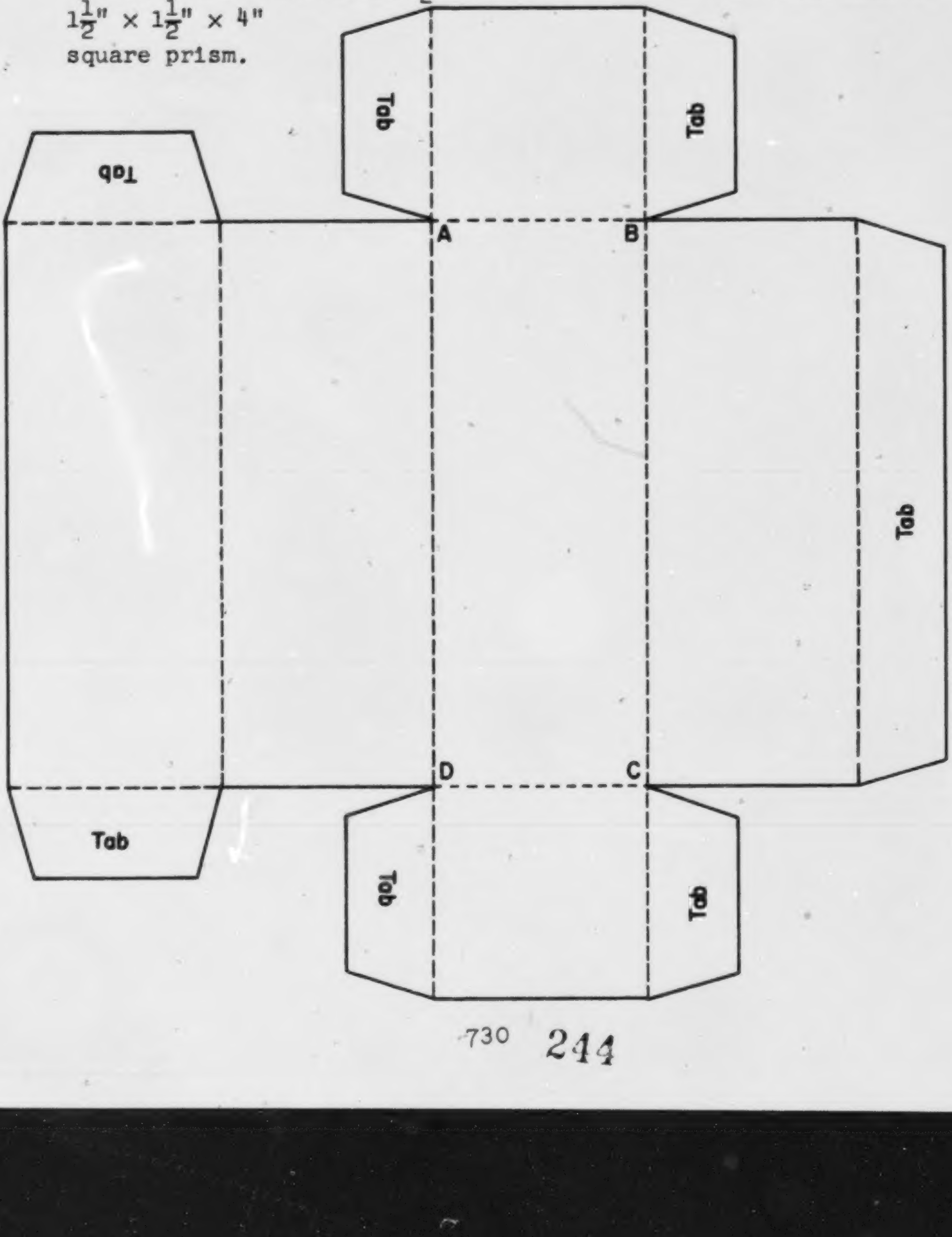
A prism can have any polygonal region for a base but its other faces must be rectangular regions. The two bases must, of course, be parallel and congruent. The bases are also faces. If the polygon outlining the base is a rectangle, the prism is a rectangular prism. If the polygon is a triangle, the prism is a triangular prism; if it is a hexagon, it is a hexagonal prism, etc.

Directions for making paper models of a square, rectangular, and triangular prism are given on the following pages. Every child should have some sort of a set of models for reference. Pupils may want to bring models from home rather than making paper models. Paper models are not very sturdy. It is realized that the construction of paper models is time consuming. It may well be that the teacher will want the children to do most of this work outside of class time. If the patterns for the models are duplicated and then pasted on heavier paper, such as construction paper, a stronger model can be made.

The development for the prism as contained in the Pupil Book is in sufficient detail to follow. The use of large models by the teacher will help develop the understandings of this section.

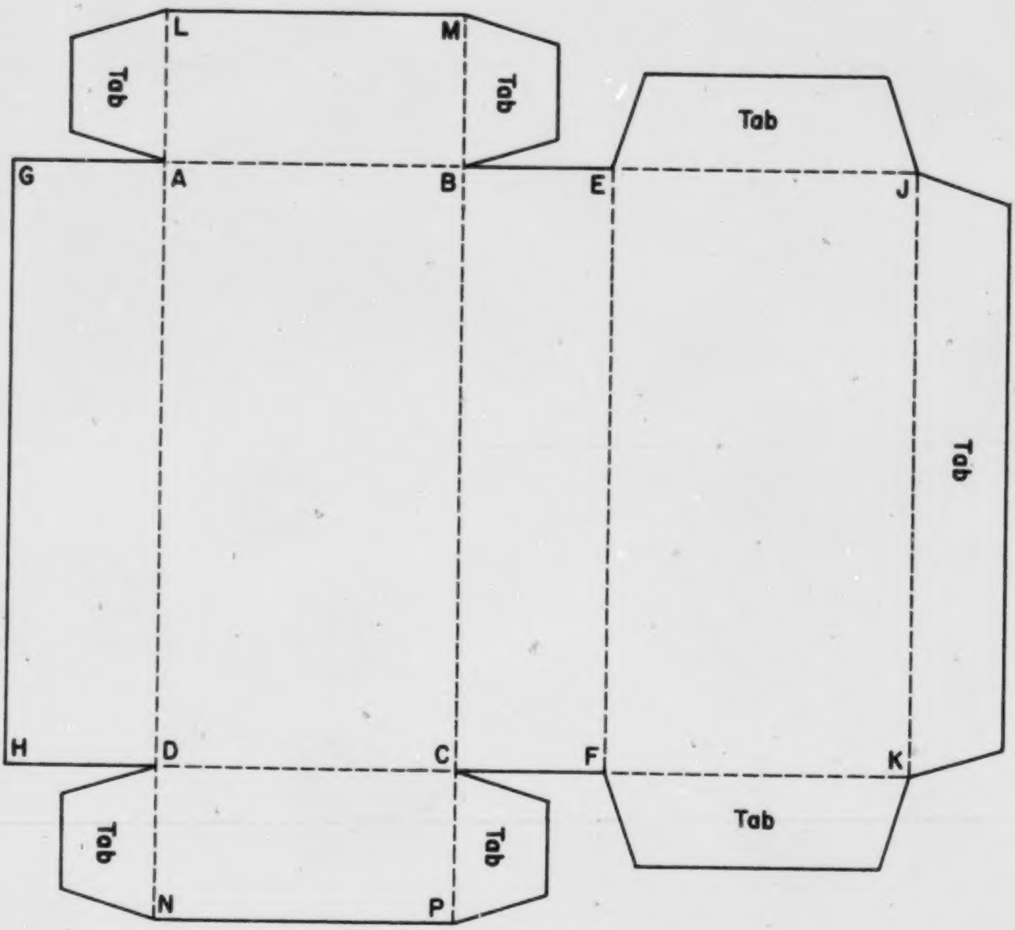
PRISM - Construction of a square prism:

1. Draw a rectangle ABCD as shown.
2. Draw, as shown, three other rectangles congruent to rectangle ABCD with tabs, as shown.
3. Draw the two squares along $\overline{AB}$ and $\overline{DC}$ with tabs, as shown.
4. Cut around the boundary of the figure and fold along the dashed line segments.
5. Use scotch tape or paste to hold the model together. The tabs will help give rigidity to the model. You may want to trim them some if you use scotch tape.
6. The bases of this rectangular prism are squares, hence the name - square prism.
7. This picture has been reduced photographically. The original had the length of $\overline{AB}$ as $1\frac{1}{2}$" and that of BC as 4". This made a $1\frac{1}{2}$" × $1\frac{1}{2}$" × 4" square prism.



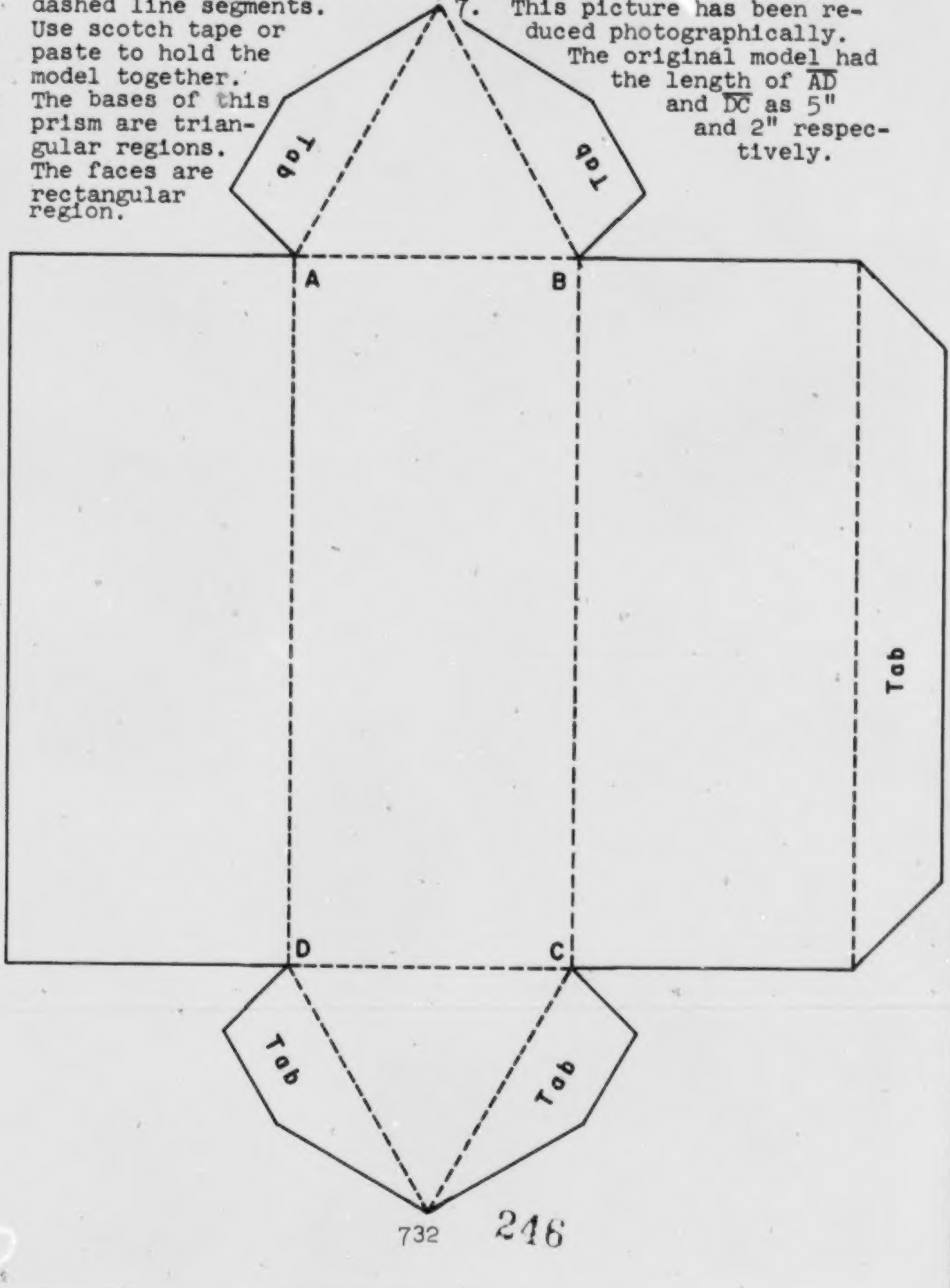
PRISM - Construction of a rectangular prism:

1. Draw a rectangle ABCD as shown.
2. Draw rectangles BEFC, AGHD, EJKF, ABML, and DCPN, with tabs, as shown.
3. Be sure that ABCD ≅ EJKF; AGHD ≅ BEFC; ABML ≅ DCPN.
4. Cut around the boundary of the figure and fold along the dashed line segments.
5. Use scotch tape or paste to hold the model together. The tabs will help strengthen the model. You may want to trim them some if you use scotch tape.
6. The bases of this rectangular prism are rectangular regions.
7. This picture has been reduced photographically. The original model had the lengths of $\overline{HD}$, $\overline{DC}$, $\overline{CF}$, $\overline{FK}$, $\overline{CP}$, and $\overline{CB}$ as 1", 2", 1", 2", 1", and 4", respectively.



PRISM - Construction of a triangular prism:

1. Draw a rectangle ABCD as shown.
2. Draw two other rectangles congruent to rectangle ABCD with tab, as shown.
3. Draw two equilateral triangles with tabs, as shown.
4. Cut around the boundary of the figure and fold along the dashed line segments.
5. Use scotch tape or paste to hold the model together.
6. The bases of this prism are triangular regions. The faces are rectangular region.
7. This picture has been reduced photographically. The original model had the length of $\overline{AD}$ and $\overline{DC}$ as 5" and 2" respectively.



Pyramid

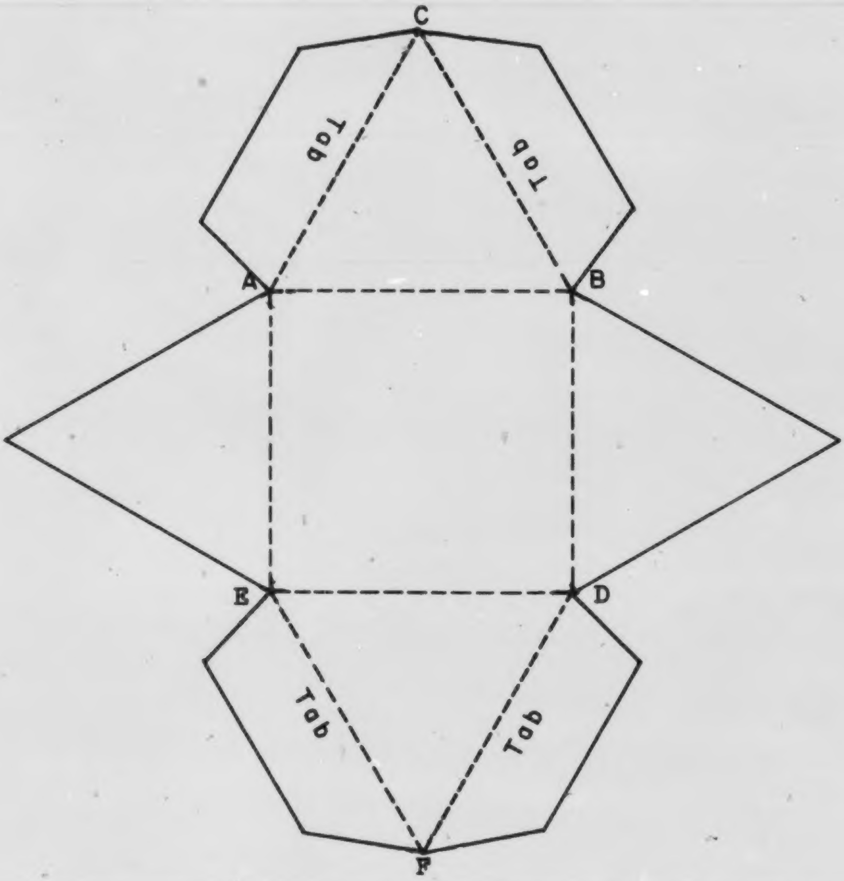
A discussion of the pyramid is given in the mathematical background section of this unit.

Many children will already have a general idea of pyramid. The mathematical notion of pyramid will need to be develeloped.

The pyramid is "hollow" - we think only of the faces and not the points which are on its "inside." A pyramid can have any polygonal region for its base. Its other faces are always triangular regions. The pyramid is named by the shape of its base. Thus, we have square pyramids, rectangular pyramids, triangular pyramids, hexagonal pyramids, and so on.

Patterns for making a square pyramid and a rectangular pyramid are given on the following pages. A pattern for a special kind of pyramid, a tetrahedron, is also given.

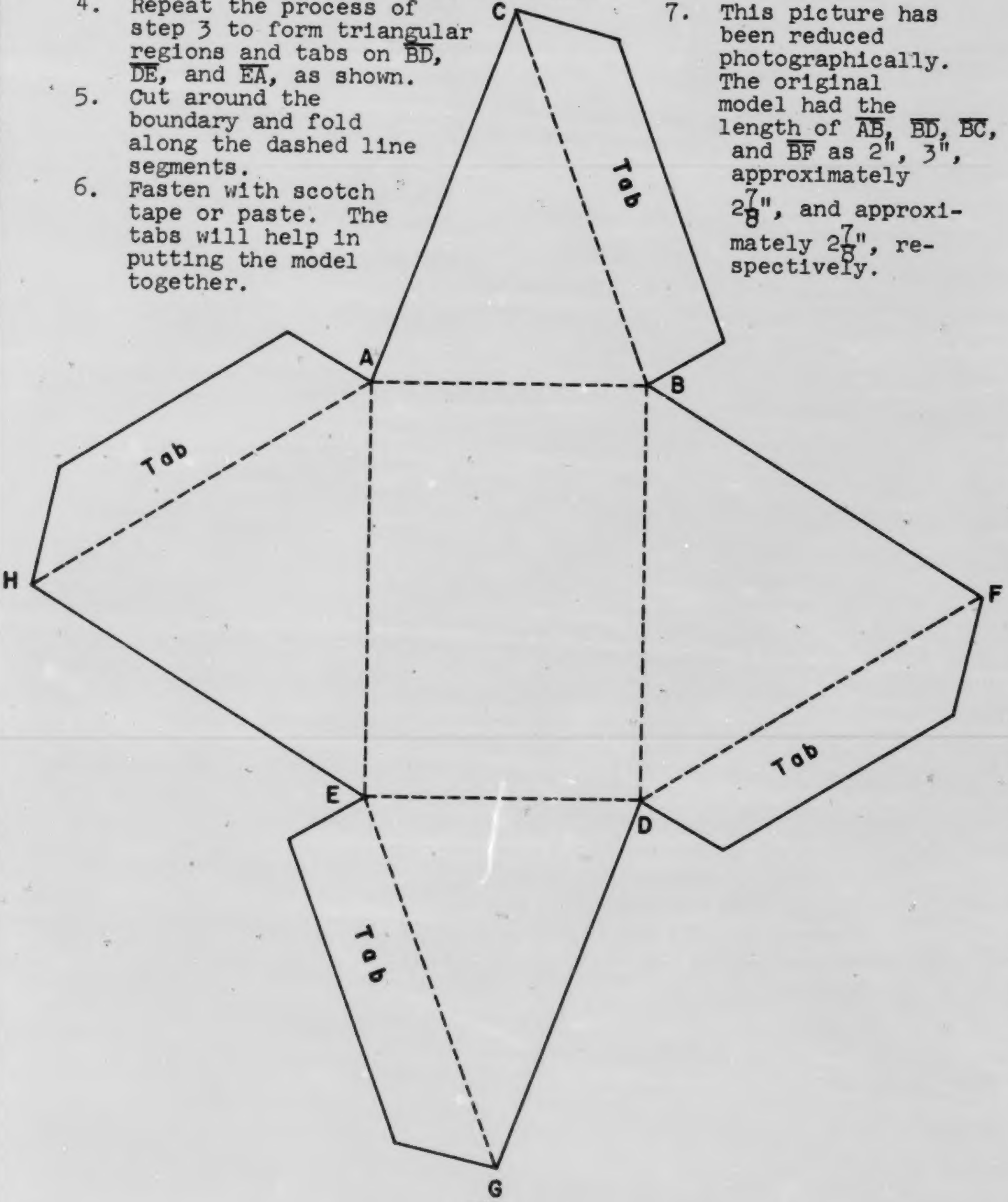
PYRAMID - Construction of a square pyramid.



1. Draw a square ABDE as shown.
2. Draw the arcs with centers at A and B and radius $\overline{AB}$. Label the intersection shown as C.
3. Draw dashed line segments $\overline{AC}$ and $\overline{BC}$ to form "dashed" equilateral triangle ABC. Draw tabs as shown.
4. Repeat step 3 to obtain "dashed" equilateral triangle EDF with tabs, as shown.
5. Draw the equilateral triangle shown on $\overline{BD}$ and $\overline{AE}$.
6. Cut around the boundary and fold along the dashed line segments.
7. Fasten with scotch tape or paste. The tabs will help in putting the model together. You may want to trim some of them if you use scotch tape.
8. This picture has been reduced photographically. The original model had the lengths of $\overline{AB}$ as 2".

PYRAMID - Construction of a rectangular pyramid:

1. Draw "dashed" rectangle ABDE as shown.
2. Draw the arcs with centers at A and B and radius $\overline{AC}$. Label the intersection shown as C.
3. Draw $\overline{AC}$ and dashed line segment BC to form isosceles triangle ABC. Draw tab, as shown.
4. Repeat the process of step 3 to form triangular regions and tabs on $\overline{BD}$, $\overline{DE}$, and $\overline{EA}$, as shown.
5. Cut around the boundary and fold along the dashed line segments.
6. Fasten with scotch tape or paste. The tabs will help in putting the model together.
7. This picture has been reduced photographically. The original model had the length of $\overline{AB}$, $\overline{BD}$, $\overline{BC}$, and $\overline{BF}$ as 2", 3", approximately 2$\frac{7}{8}$", and approximately 2$\frac{7}{8}$", respectively.



CYLINDER

A discussion of the cylinder is given in the mathematical background of this unit.

Some children may already have some concept of a cylinder. Probably their idea is of a special kind of cylinder, the circular cylinder. Cylinders can have bases that are not circular regions. For example, some oil tanks for homes, gasoline tanks for cars, and toothbrush containers are models of cylinders but are not models of circular cylinders.

Children can find many models of circular cylinders such as frozen juice cans, baby food cans, fruit and vegetable cans, and oatmeal boxes. It must be remembered, however, that a cylinder has "both ends covered." A cylinder has two bases which are congruent and parallel, and a cylindrical surface.

Patterns for making a cylinder and a circular cylinder are given on the following pages.

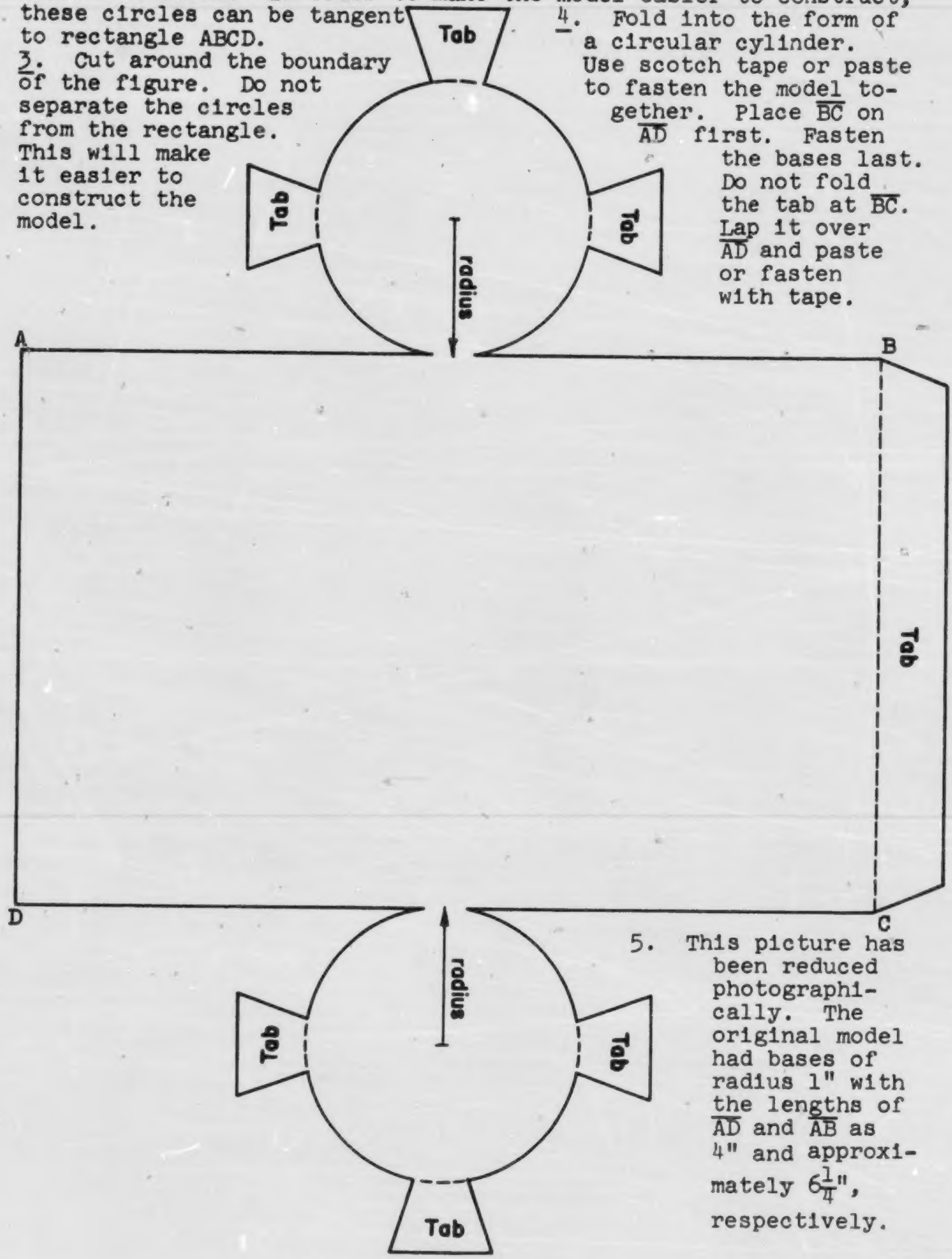
The demonstration in Exercise 1 of Thinking Together will help children get a clearer idea of a cylinder. An oatmeal box would be a good model to use. To get just the idea that the curved surface of a cylinder is a rectangular region when it is straightened out and placed in one plane, cut the wrapper off any cylindrical can. Be sure to make your cut perpendicular to the top and bottom edge of the wrapper to obtain the rectangular region. If the cut is not perpendicular, the outline of the resulting figure will be a parallelogram.

CYLINDER - Construction of a circular cylinder

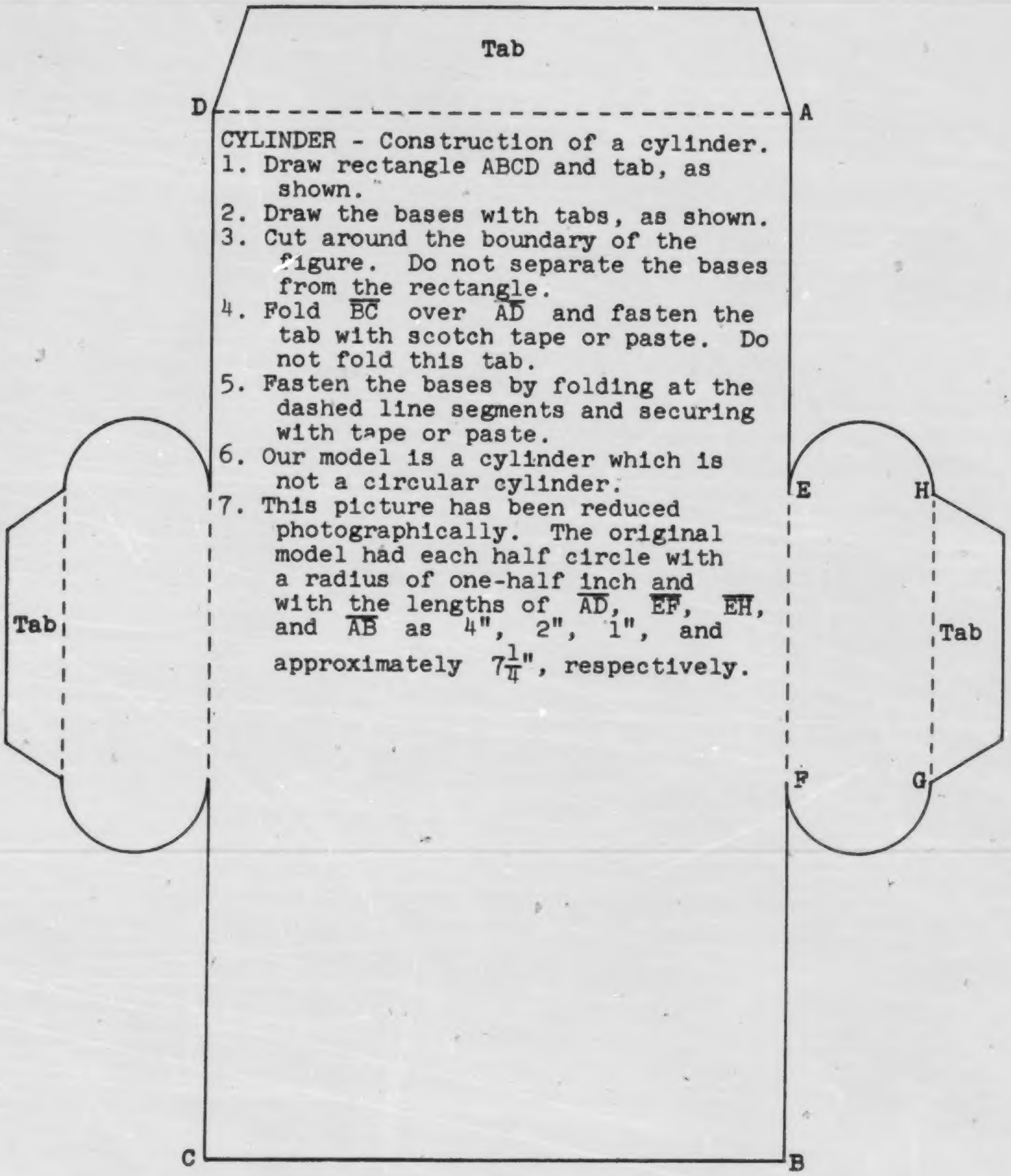
1. Draw rectangle ABCD. 2. Draw two congruent circles with radius as shown. In order to make the model easier to construct, these circles can be tangent to rectangle ABCD.

3. Cut around the boundary of the figure. Do not separate the circles from the rectangle. This will make it easier to construct the model.

4. Fold into the form of a circular cylinder. Use scotch tape or paste to fasten the model together. Place $\overline{BC}$ on $\overline{AD}$ first. Fasten the bases last. Do not fold the tab at $\overline{BC}$. Lap it over $\overline{AD}$ and paste or fasten with tape.



5. This picture has been reduced photographically. The original model had bases of radius 1" with the lengths of $\overline{AD}$ and $\overline{AB}$ as 4" and approximately $6\frac{1}{4}$", respectively.



CYLINDER - Construction of a cylinder.

1. Draw rectangle ABCD and tab, as shown.
2. Draw the bases with tabs, as shown.
3. Cut around the boundary of the figure. Do not separate the bases from the rectangle.
4. Fold $\overline{BC}$ over $\overline{AD}$ and fasten the tab with scotch tape or paste. Do not fold this tab.
5. Fasten the bases by folding at the dashed line segments and securing with tape or paste.
6. Our model is a cylinder which is not a circular cylinder.
7. This picture has been reduced photographically. The original model had each half circle with a radius of one-half inch and with the lengths of $\overline{AD}$, $\overline{EF}$, $\overline{EH}$, and $\overline{AB}$ as 4", 2", 1", and approximately $7\frac{1}{4}$", respectively.

CONE

A discussion of the cone is given in the mathematical background section of this unit.

Construct a model of a cone as as shown on the following page. It is not necessary to use only a semicircular region for the curved surface of the cone. Any part of a circle and its interior could be used. The larger the part used, the shorter will be the resulting cone. The semicircular region is a handy part to use and makes a "good" model, so that is the pattern given here.

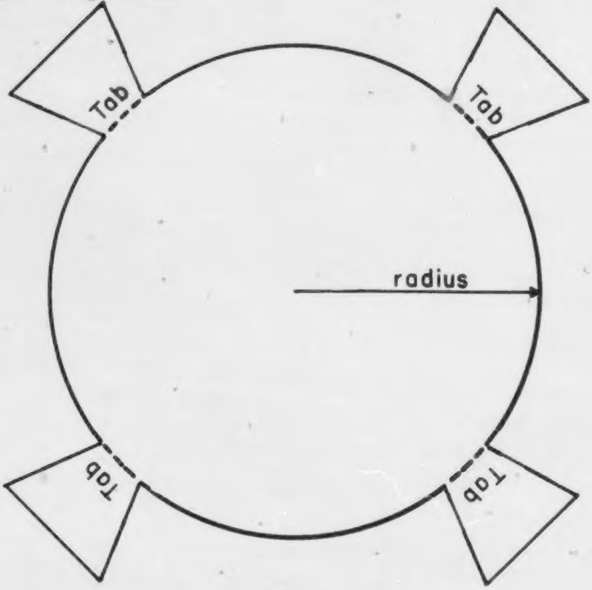
After Exercise 3 in Thinking Together (in the section on the cone) has been completed, then disassemble the cone. Remove the circular part first, leaving an exposed circular edge. The model now resembles an ice cream cone without the ice cream. Next cut the remaining surface along a line segment from the circular edge to the vertex. Lay this part flat against the chalkboard. It should look like a piece of pie which suggests part of the union of a circle and its interior. Ask the children what this looks like. Then reconstruct the cone.

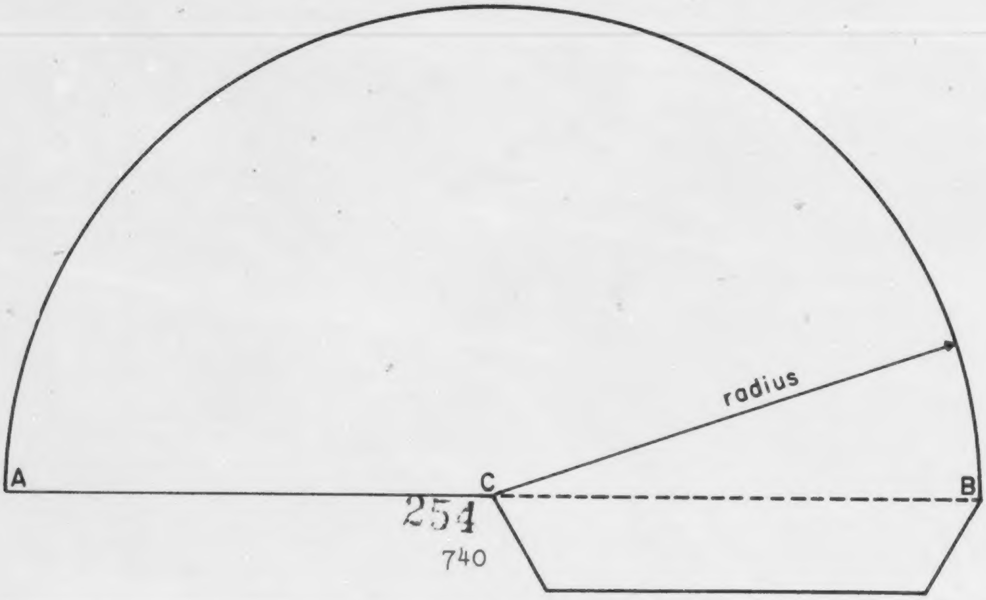
A cone may be thought of in another way. Start with a circle and its interior. Choose a point directly over the center of the circle not in the plane containing the circle. This point will be the vertex of the cone. Think of all line segments with one endpoint, the vertex and the other endpoint on the circle. The union of all these segments and the circle with its interior is the cone. The ribbons from a Maypole suggest the segments from the vertex to the circular edge.

In Exercise 4, to emphasize the fact that the intersection described is a triangle, the teacher should hold the model of a cone at the tip and at the center of the circle so that the line segment joining the tip and the center of the circle is horizontal. Look at the shadow produced when a light is held directly over the figure. The outline of the shadow will represent a triangle. If the model is held so that the line segment is vertical, the shadow will represent a circle and its interior. Name objects which suggest cones or parts of cones.

CONE - Construction of a cone

1. Use a compass to draw a circle with a radius as shown in the diagram. Draw tabs as shown.
2. Cut around the boundary of this figure. The circular region will be the base of the cone.
3. Use a compass to draw a semicircle with a radius as shown in in the diagram. C is the center of the circle. $\overline{AB}$ is a diameter. Draw the tab as shown.
4. Cut around this figure.
5. Fasten $\overline{AC}$ to $\overline{BC}$ with scotch tape or paste so that $\overline{AC}$ falls on $\overline{BC}$.
6. Fasten the base to this model by folding the tabs and using scotch tape or paste.





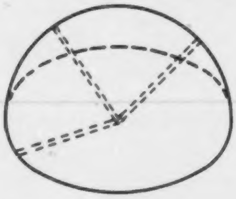
SPHERE

A discussion of the sphere is given in the mathematical background section of this unit.

The mathematical idea of a sphere will be rather difficult for some children to grasp. Certainly if we should begin our study of sphere by saying, "A sphere is the set of all the points in space that are the same distance from a chosen point called the center" some children would be quite confused. Yet this is the idea we want children to have at the end of this brief section on the sphere.

This lesson might be started by identifying a hollow rubber ball as a model of a sphere. Emphasize that the model of the sphere is just the "covering" of the ball and does not include its interior.

Exercise 1 in Thinking Together could be performed with this rubber ball and a flashlight or some other source of light. For Exercises 2 and 3, actually cut the hollow ball into two congruent sections to show that the edges represent a circle. (Some teachers have used a globe that can be separated into two sections to show this.) Children might indicate where they think the center of the sphere would be. A stick or piece of wire could be used to help develop the idea of sphere. This stick could be held so that one end is at the center of the sphere and the other end moved about one of the half-spheres to show that the end moved about is always touching the "covering" of the ball. The sketch indicates how this might be done. Exercise 6 should help develop this idea, too. It might prove helpful to use an orange for a model of a sphere. The inside of the orange is not part of the sphere. By cutting the orange, as we suggested for the rubber ball, and then removing the fruit, the same concepts could be developed.



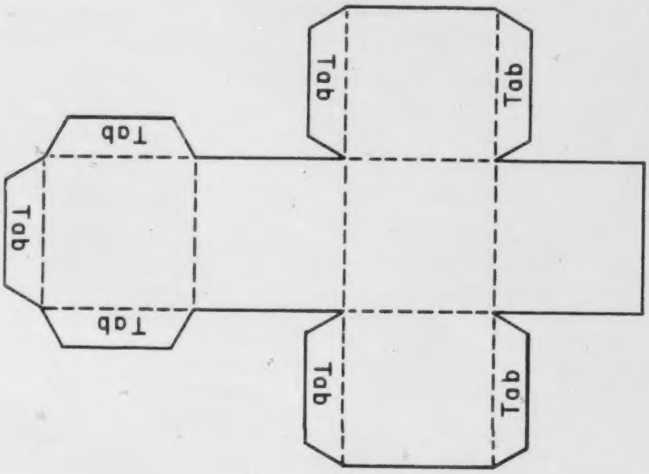
A small piece of clay or styrofoam could be used to represent a center of a sphere. Many toothpicks could be stuck in it at random. The other ends of the toothpicks would then represent points of a sphere. Children could see that the ends of the toothpicks do describe a sphere.

Exercises 7, 8, and 9 ask the pupil to verbalize his idea of a sphere. Some children may have difficulty doing this.

CUBE

Patterns for the construction of models of a cube and tetrahedron are given to develop the ideas given in the exercises. Have several of these paper models or other geometric models available for exploratory use.

Construction of a cube.



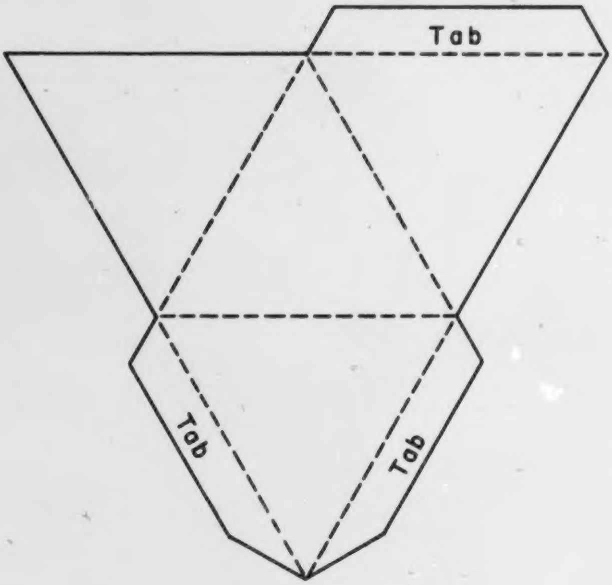
1. Draw six squares (at least 4" x 4") on heavy paper or tagboard as shown in the figure above.

2. Cut around the boundary of the figure and fold along the dotted lines.

3. Use scotch tape or paste to fasten the model together. The tabs will help give strength to the model.

4. The model of a cube has six faces, eight vertices, and twelve edges.

TETRAHEDRON



1. Draw four equilateral triangles (with the given segment at least 4" long) as shown in the figure. Refer to Page 714 for construction of an equilateral triangle.

2. Cut around the boundary of the figure and fold along the dotted lines.

3. Use scotch tape or paste to fasten the model together. The tabs will help make the model more rigid.

4. The model of a tetrahedron has four faces, four vertices, and six edges.

Thinking Together

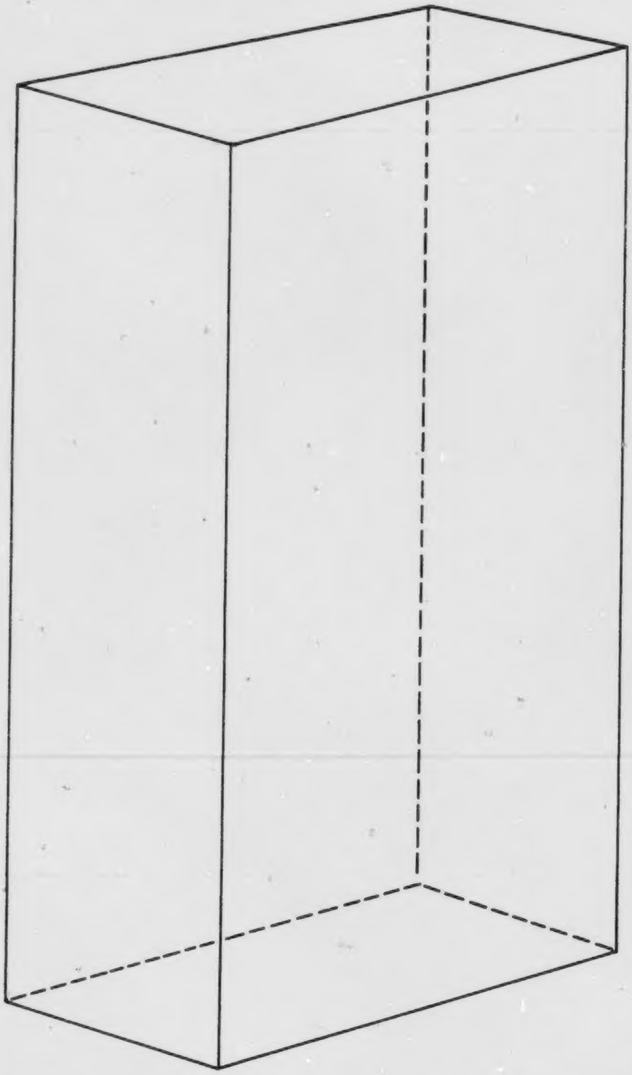
We are going to look at some objects. The surfaces of these objects represent sets of points in space. These sets have names which you will find below the pictures of the objects.

The objects are called models because they represent sets of points. Parts of the surface of some of these objects remind us of parts of planes because they are flat.

A closer look at these flat parts shows that one suggests a triangular region (a triangle and its interior). Another flat part of a model suggests the union of a quadrilateral and its interior. Still others remind us of circular regions (circles and their interiors).

Not all parts of the surfaces are flat. For example, the sphere, the cylinder, and the cone have parts of surfaces which are not flat.

P438



Rectangular Prism

Rectangular Prism

Thinking Together

Let us look closely at this model of a rectangular prism. Any one of the flat parts of the surface suggests a quadrilateral and its interior. The set of points on the flat part is called a face. The union of all the faces is called a rectangular prism. A rectangular prism consists of the entire surface of the model but not the inside of the model.

1. Look at a face.

   a) Trace with your finger the edge of a face. The edges represent the sides of what figure? (a quadrilateral or a rectangle)

   b) Show with your finger the vertices of the rectangle.

   c) Mark with your pencil three points in the interior of the rectangle.

2. How many faces does the rectangular prism have? (6) The edges of the model represent line segments. These line segments are the intersection of two different faces and are called edges of the rectangular prism.

3. a) Trace an edge with your finger.
   b) Show two faces whose intersection is this edge.
   c) Count the number of edges of the rectangular prism. How many are there? (12)

The corners of the model represent points. Each such point is called a *vertex* of the prism; it is also a vertex of each of the three quadrilaterals which come together at the corner. The plural of the word "vertex" is "vertices", so we can speak of one vertex and several vertices.

4. a) Mark a vertex of the rectangular prism on your model with a pencil.
   b) What three quadrilaterals have this point as a vertex?
   c) How many vertices does the rectangular prism have? (8)

5. Name some other objects which represent a rectangular prism. (chalk box, crayon box, book)

6. If a wooden block were hollow, would it still represent a rectangular prism? (Yes, in fact *only* a hollow block really represents a rectangular prism. The prism is the union of the quadrilateral regions. It does not include points "inside" the block.)

Exercise Set 2

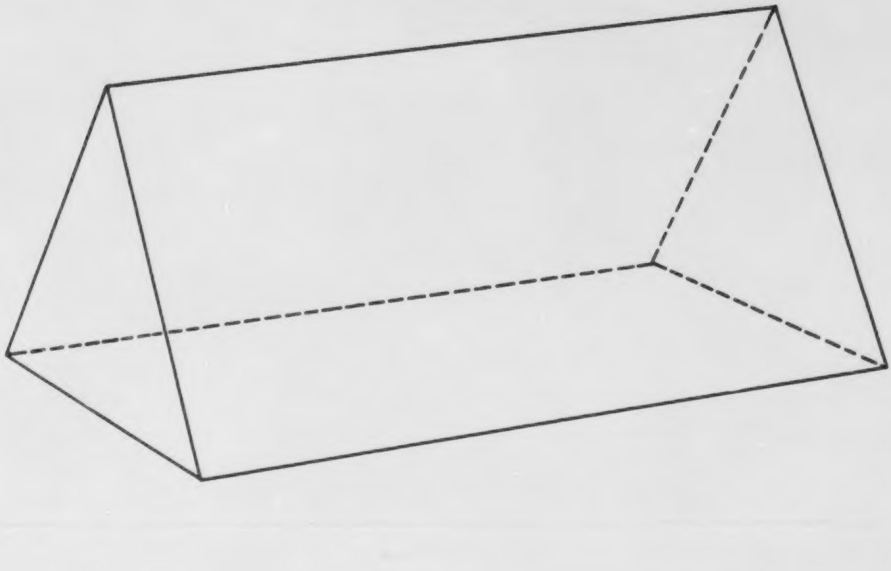
. 1. Complete the following sentences, using a separate sheet of paper.

a) A rectangular prism has six faces.

Each face represents a rectangular region.

b) The rectangular prism has twelve edges and eight vertices.

2. Is a rectangular prism hollow? (yes)



Triangular Prism

Triangular Prism

Thinking Together

The model we will study next represents a triangular prism. As in the rectangular prism, the flat parts of the model will represent plane regions which are called faces. The triangular prism is the union of the faces. (Notice that a triangular prism, as we have defined it, is "hollow".)

1. Are all the faces of the triangular prism the union of rectangular regions? (no)
2. Indicate a face which does not represent a rectangular region. (Point to a triangular face.)
3. What does this face represent? (a triangle and its interior.) We call a face which represents a triangle and its interior a triangular face or a triangular region.
4. How many triangular faces are there? (two)
5. How many rectangular faces are there? (three)

The triangular prism also has line segments which are the intersections of two faces and are called edges. The endpoints of the edges are called vertices of the prism. They are represented by the corners of the model.

6. How many edges does a triangular prism have? (nine)
7. How many vertices does a triangular prism have? (six)
8. Name some objects which represent a triangular prism. (pup-tent, folded paper)

P444

Exercise Set 3

1. Look at your model and write on a separate piece of paper the words that complete the following sentences.

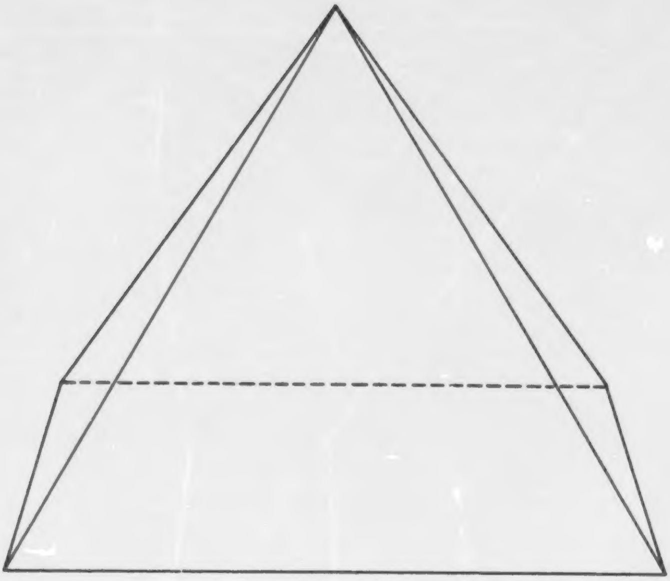
   a) A triangular prism has 2 triangular faces.

   b) A triangular prism has 5 faces.

   c) A triangular prism has 9 edges.

   d) A triangular prism has 6 vertices.

P446



Pyramid

266

Pyramid

Thinking Together

Look at the model of the pyramid pictured on page 446. It is made up of flat parts only, like those of a prism. These suggest parts of planes, and as in the case of the prism, they will be called faces. The pyramid is the union of these faces. (Notice that a pyramid, as we have defined it, is "hollow".) Faces next to each other intersect in a line segment which has endpoints called vertices.

1. a) How many triangular faces does this pyramid have? (4)
   b) Are there any faces on this pyramid which are not triangular? (yes) Trace this face with your finger.
   c) How many faces are there on this pyramid? (5)
   d) How many vertices are there on this pyramid? (5)
   e) Put your finger on a vertex which is the intersection of three faces.
   f) Show a vertex which is the intersection of four faces.
   g) How many edges does this pyramid have? (8)
   h) Trace with your finger four edges which intersect at a vertex.

2. Name some objects which suggest a pyramid.
   (some tents, Egyptian pyramids, some roofs of houses)

Cylinder

Thinking Together

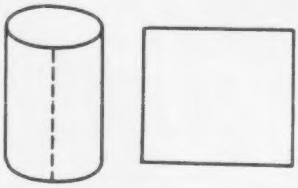
Let us look at the model of the cylinder which you have brought to class.

1. a) Are all parts of the surface of the model flat? (no)

   b) What do the flat parts represent? (a circle and its interior, sometimes called a disk or circular region]

   c) Are the circular regions the same size? (yes)

Remove the top and bottom of a box which is the model of a cylinder. Make one straight cut as shown in the picture. Spread out the part just cut so that it lies flat on the desk.



2. a) What geometric figure is represented? (rectangle)

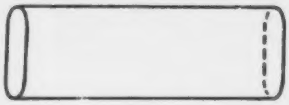
   b) Does this suggest how you might construct a model of a cylinder?

   c) Put the parts of the box together again so that it is a model of a cylinder.

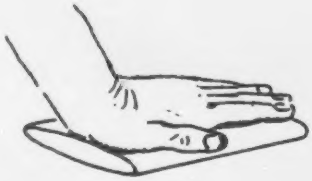
3. Name some objects which represent cylinders or parts of cylinders. (a drum, a tank, ice cream carton, food cans, some drinking glasses, some garbage cans, water towers, oatmeal boxes, drinking straws, pipelines)

4. Not all cylinders are like the oatmeal boxes or the cans that you brought. Your models are examples of a special kind of cylinder called a circular cylinder. Circular cylinders have circular regions for bases.

Many cylinders do not have circular regions for bases. Look at this picture of a cylinder. Its bases are not circular regions.



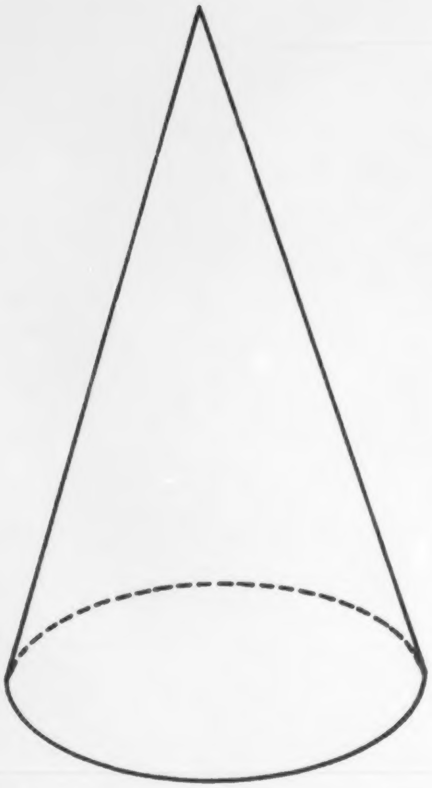
Take a can which is a model of a circular cylinder. Take out both bases. Press it down like this:



Think of bases that are oval regions being put on the can. Now we have a model of a cylinder which is not a circular cylinder.

Try to find a model of a cylinder which is not a circular cylinder. Sometimes toothbrushes come in this kind of container. Would part of the front or the back fork of a bicycle be an example of a cylinder?

P452



Cone

760 271

Thinking Together

We can see from the photographs that the model of a cone is different from that of a cylinder.

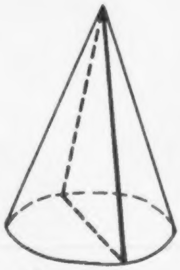
1. Describe ways in which the model of the cone differs from that of the cylinder.

2. a) What geometric figure does the flat part suggest? (a circular region)

   b) Place the model of the cone so that the flat part is on a sheet of paper and trace around the edge. What is represented? (a circle)

3. What is represented by the tip of a model of a cone? (a point) The point represented by the tip is called the vertex of the cone.

4. Suppose the model of a cone is cut in half so that the cut passes through the vertex and divides the circular region into parts of equal size.



   What will the cut edge represent? (a triangle) The cut edge is shown as the heavy line segments in this drawing.

5. Name other objects which suggest cones or parts of cones. (ice cream cones without ice cream, some paper drinking cups, an Indian tepee, the sharpened portion of a pencil.)

6. Is a cone hollow? (yes, as we have defined it.)

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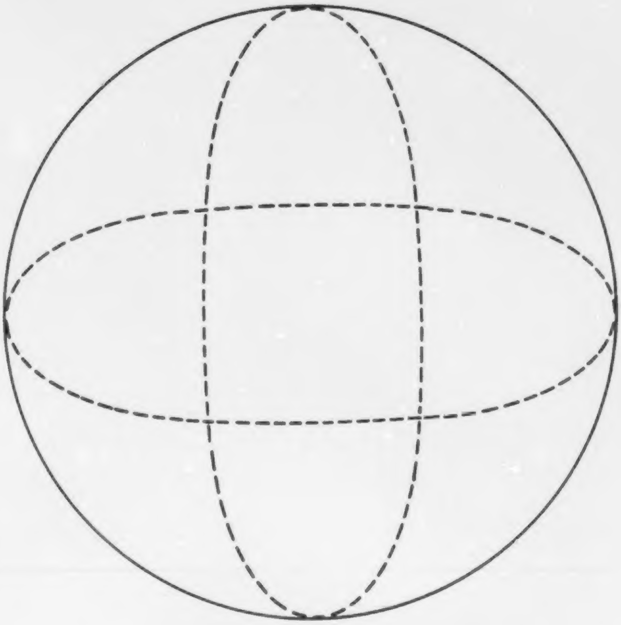


Sphere

Sphere

Thinking Together

The surface of a ball or a globe is a model of a sphere.

1. Look at the shadow of a ball on the sidewalk when the sun is directly overhead, or look at the shadow made by a ball with a light directly over it.

   What does the edge of the shadow look like? (a circle)

2. Cut a model of a sphere into two pieces of equal size.

   What geometrical figure do the edges of each half of the sphere represent? (a circle)

3. Place one of the halves of the sphere on a flat surface so that the cut edge is against the flat surface. Trace around this edge.

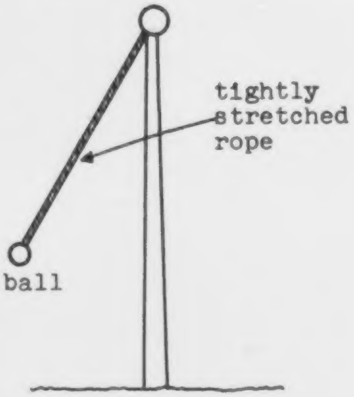
   What does the drawing represent? (a circle)

4. Do you think that all the points of the half sphere are the same distance from the center of the circle that is represented by the edge? (yes)

5. Does this suggest that there is a point inside the sphere that we might call the center? (yes)

6. Think of a tetherball. When the rope is tightly stretched, the ball is always the same distance from the point where the rope is attached.

As the ball moves around into different positions, do you think the point where the rope is attached to the ball goes through points on a sphere? (yes)



7. How would you describe a sphere?

8. Would the description of the sphere as "the set of all points in space that are the same distance from a chosen point called the center" be a good description? (yes)

9. How does this description differ from that of a circle?
(The description is the same, except that for a circle all the points must lie in the same plane.)

10. Is a sphere hollow? (yes)

EDGES AND FACES

Thinking Together

1. Do any edges of a triangular prism form a rectangle? (yes) Check with your right angle model.

2. Do any edges of a pyramid form a rectangle? (yes, if the pyramid is a rectangular pyramid.) Check with your right angle model.

3. How many faces of the rectangular prism have edges which form a rectangle? (6)

4. How many faces of the triangular prism have edges which form a rectangle? (3)

5. How many faces of the pyramid have edges which form a rectangle? (one)

6. Look at the models of a rectangular prism, a triangular prism, and a pyramid. Copy the following table on a sheet of paper and fill in the blank spaces.

	Number of Faces	Number of Vertices	Number of Edges	Number of faces plus number of vertices minus number of edges
Rectangular prism	6	8	12	2
Triangular prism	5	6	9	2
Pyramid	5	5	8	2

a) Add the number of faces and the number of vertices of the rectangular prism. Subtract the number of edges from this sum. What is your answer? (2)

b) Do the same for the triangular prism and the pyramid. What is your answer in each case? (2)

c) Look at other geometric figures made up of portions of planes. Is the number of faces plus the number of vertices minus the number of edges the same for these geometric figures ? (yes)

d) Do you think you will always get the same answer if you add the number of faces and the number of vertices, then subtract the number of edges of any geometric figure which is the union of parts of planes? (yes)

e) Write a mathematical sentence for your answer of 6d. (Answer will vary. It should be $F + V - E = 2$ where F, V, and E are names for the number of faces, vertices, and edges, respectively.)

Chapter 9

LINEAR MEASUREMENT

PURPOSE OF UNIT

The purpose of the unit is to introduce the pupil to the concept of length and to the technique of linear measurement.

Our study of linear measurement may be divided into four major stages which parallel the historical development:

1. Intuitive awareness of difference in size
2. Choice of an arbitrary unit, understanding that the unit must be of the same nature as the thing to be measured - a line segment to measure a line segment
3. Selection of standard units for purposes of communication
4. Devising a suitable scale for convenience in measuring

MATHEMATICAL BACKGROUND

Long before the child comes to school he has experience with the notion of measure. His father is taller than he is; his sister is younger than he is; he gained three pounds since his last medical check-up; the new house is bigger than the old house. By the time the child has come to this unit, towards the end of the fourth grade, he understands and makes such statements as, "My Dad is 6 feet two," "The milkman leaves three quarts of milk for us every day," "It takes me 15 minutes to go to school." He knows that there are many different kinds of measures: time, weight, liquid, linear; and he applies the appropriate units. It is the purpose of the present Unit not only to expand the child's knowledge of linear measure, but also to deepen his intuitive understanding.

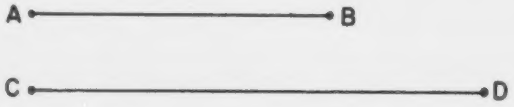
It is of interest to note that in the development of the idea of measurement for the pupil, we are actually following the historical development of this concept. The counting of discrete, separate objects (like the number of sheep in a herd) was not a technique applicable to measuring a continuous curve (like a side of a wheat field). At first the notion of size was realized intuitively. One boundary was longer than another; one piece of land was larger than another. Later when fields bordered more closely on each other, more refined measures were necessary. When a unit of measure (e.g. that part of a rope between two knots) was agreed upon, it was possible to designate a piece of property as having a length of "50 units of rope" and having a width of "30 units of rope." With the increase in travel and communication, it became obvious that "50 units of rope" did not represent the same length to people not familiar with the unit. The need for a standard unit was realized. Once a standard unit was accepted, a scale, for greater convenience in measuring, was devised.

Comparison and Length

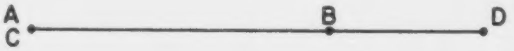
A basic concept in measurement is the difference between comparisons of the sizes of collections of discrete objects that can be counted, and comparisons of the sizes of objects, like line segments, that are "continuous" so that counting is not directly applicable. It is meaningful to count the number of objects when comparing the sizes of collections of marbles, or of eggs, or of chairs in a room, or of pages in books. There is a clarity of intent in a statement of comparison such as, "You have a larger collection of recordings than I have. I counted 148 records in your albums, while I have only 110 in mine."

But if the sizes of two line segments are to be compared, what could we *count*? Surely not the number of points in the sets which constitute the line segments, for there are more points on any line segment than we can count in our number system. Clearly, then, *counting* points is not a feasible procedure for comparing the sizes of two line segments.

However, we can always conceive of comparing two line segments, for example, $\overline{AB}$ and $\overline{CD}$,



as to size by laying off $\overline{AB}$ along $\overline{CD}$ in such a way that the points A and C coincide:



In the example illustrated above, it turns out that B then falls between C and D. When this happens, we say that $\overline{AB}$ is *shorter* than $\overline{CD}$ or that $\overline{AB}$ is of *smaller length* than $\overline{CD}$. For some line segments, $\overline{AB}$ and $\overline{CD}$, it might happen that B coincides with D, in which case we would say that $\overline{AB}$ is of the *same length* as $\overline{CD}$. (Recall that in this case we also say that $\overline{AB}$ and $\overline{CD}$ are *congruent*.) Or it might happen that D falls between A and B, in which case we would say that $\overline{AB}$ is *longer* than $\overline{CD}$ or that $\overline{AB}$ is of *greater length* than $\overline{CD}$.

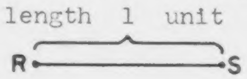
When, as in the illustrations on page 769, the line segments, $\overline{AB}$ and $\overline{CD}$, can be conveniently drawn on a sheet of paper, this comparison can be carried out, approximately, by making a tracing of $\overline{AB}$ and placing it on top of the drawing of $\overline{CD}$. But even if the line segments, $\overline{AB}$ and $\overline{CD}$, were much too long or much too microscopically short to be drawn satisfactorily on a sheet of paper at all, we would still conceive of $\overline{AB}$ and $\overline{CD}$ as being such that exactly one of the following three statements is true:

(1) $\overline{AB}$ is shorter than $\overline{CD}$.
(2) $\overline{AB}$ is of the same length as $\overline{CD}$.
(3) $\overline{AB}$ is longer than $\overline{CD}$.

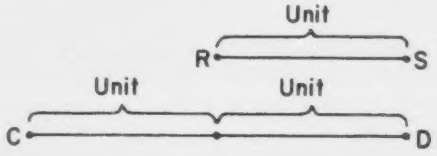
Recall (from Chapter five) that in mathematics we think of the endpoints A and B of any given line segment AB as being *exact locations in space*, although these endpoints can be represented only approximately, by chalked or penciled dots, etc. In the same way we think of the line segment, $\overline{AB}$, as having a certain *exact length*, although this length can be determined only approximately, by measuring a chalk or pencil drawing representing $\overline{AB}$.

Units and Measurements

Let us describe this process of measurement more closely. The first step is to choose a line segment, $\overline{RS}$, to serve as *unit*. This means that we select $\overline{RS}$ and agree to consider its length to be described or measured exactly by the number 1.

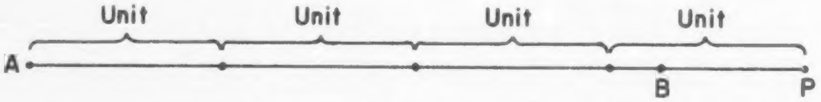


Now we can conceive of a line segment, $\overline{CD}$, such that the unit $\overline{RS}$ can be laid off exactly twice along $\overline{CD}$, as suggested in the picture below.



We say that $\overline{CD}$ has length exactly 2 units, although $\overline{CD}$ can be represented only approximately by a drawing. In the same way we can conceive of line segments of length exactly 3 units, or exactly 4 units, or exactly any larger number of units, although such line segments can be drawn only approximately. In fact, if a line segment is very long -- say a million inches long -- we wouldn't want to try to draw it even approximately; but we can still think about such a line segment.

We can also conceive of a line segment, $\overline{AB}$, such that the unit $\overline{RS}$ will not "fit into" $\overline{AB}$ a whole number of times at all. In the picture below we have shown a line segment, $\overline{AB}$,



such that starting at A the unit can be laid off 3 times along $\overline{AB}$ without quite reaching B, though if we were to lay off the unit 4 times we would arrive at a point P which is well beyond B. What can we say about the length of $\overline{AB}$? Well, we can surely say that $\overline{AB}$ has length greater than 3 units and less than 4 units. In this particular case, we can also estimate visually that the length of $\overline{AB}$ is nearer to 3 units than to 4 units, so we can say that to the nearest unit $\overline{AB}$ has length 3 units. This is the best we can do without considering fractional parts of units, or else shifting to a smaller unit.

Let us emphasize one thing about terminology. In a phrase like "length 3 units" we refer to the number 3 as the "measure 3." The point here is simply to have a way of referring to the numbers involved so that we can add them, etc. Remember that we have learned how to apply arithmetic operations like addition only to numbers. We don't add yards any more than we add apples. If we have 3 apples and 2 apples, we have 5 apples altogether, because

$$3 + 2 = 5.$$

Likewise if we have 3 yards of ribbon and 2 more yards of ribbon, we have 5 yards of ribbon altogether, again because

$$3 + 2 = 5.$$

Standard Units and Systems of Measures

The acceptance of a standard unit for purposes of communication is soon followed by an appreciation of the convenience of having a variety of standard units. An inch is a suitable standard unit for measuring the edge of a sheet of paper, but hardly satisfactory for finding the length of the school corridor. While a yard is a satisfactory standard for measuring the school corridor, it would not be a sensible unit for finding the distance between Chicago and Philadelphia.

The standard units of linear measure: inch, foot, yard, and mile, which we treat in this chapter, are units in the British-American System of Measures. In the eighteenth century in France, a group of scientists developed the system of measures which is known as the Metric System.

There are two things to be noted in advance about the brief discussion of the metric system which follows. First, only metric units of length are discussed here. To be sure, there are also metric units of volume, weight, etc., but there is no need to introduce them at this point. Second, the discussion below is intended as background information for the teacher only. It is neither expected nor desired that this material will be taught in this form to pupils. The only material on the metric system intended for pupils is that which specifically appears in the pupil's edition of the unit.

In the metric system the basic unit of length is the meter, which is approximately 39.3 inches or a little more than 1 yard. Thus in the Olympic Games, where the metric system is used, we have the 100 meter dash, which is just a little longer than the 100 yard dash.

The principal advantage of the metric system over the British-American system lies in the fact that the metric system has been designed for ease of conversion between the various metric units using the decimal system of numeration. The metric unit which is just one tenth of a meter is called the decimeter. Thus

$$1 \text{ decimeter} = .1 \text{ meters},$$
$$10 \text{ decimeters} = 1 \text{ meter}.$$

The metric unit which is just one hundredth of a meter (and hence one tenth of a decimeter) is called the centimeter. Thus

$$1 \text{ centimeter} = .01 \text{ meters},$$
$$100 \text{ centimeters} = 1 \text{ meter}.$$

The metric unit which is just one thousandth of a meter (and hence one tenth of a centimeter) is called the millimeter. Thus

$$1 \text{ millimeter} = .001 \text{ meters},$$
$$1000 \text{ millimeters} = 1 \text{ meter}.$$

To see how easily conversions are made between these units, consider a length of 3.729 meters. To convert this to decimeters we need only multiply by 10, because 10 decimeters = 1 meter. Therefore, since $3.729 \times 10 = 37.29$, we see that

$$3.729 \text{ meters} = 37.29 \text{ decimeters}.$$

To convert 3.729 meters to centimeters, we need only multiply by 100, because 100 centimeters = 1 meter. Therefore, since $3.729 \times 100 = 372.9$, we see that

$$3.729 \text{ meters} = 372.9 \text{ centimeters}.$$

Finally, to convert 3.729 meters to millimeters, we need only multiply by 1000, because 1000 millimeters = 1 meter. Therefore, since $3.729 \times 1000 = 3729$, we see that

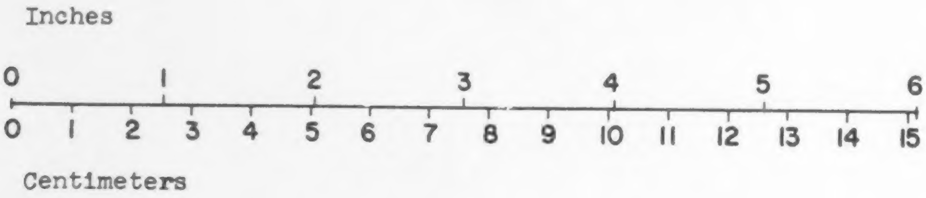
$$3.729 \text{ meters} = 3729 \text{ millimeters}.$$

Also since

$$3.729 = 3 + .7 + .02 + .009,$$

we can think of 3.729 meters as being the sum of 3 meters, 7 decimeters, 2 centimeters, and 9 millimeters. This brings out the connection between the metric units and the place values of the decimal system of numeration.

We have already noted that in the metric system the meter is the unit which corresponds approximately to the yard in the British-American system. The metric unit which most closely corresponds to the inch is the centimeter. Since 1 meter is approximately 39.3 inches, and since 1 centimeter = .01 meter, we see that 1 centimeter is approximately $39.3 \times .01$ inches or .393 inches. This is nearly .4 inches or a little less than half an inch. Below is illustrated for comparison a scale of inches and a scale of centimeters.



So far we have said nothing about metric units larger than the meter. The most useful of these is the kilometer, which is defined to be 1000 meters. The kilometer is the metric unit which most closely corresponds to the British-American mile. It turns out that one kilometer is a little more than six tenths of a mile:

1 kilometer = .6 miles (approximately).

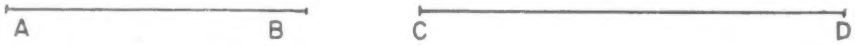
Since the metric system is the official system of measures for many countries, and is used by scientists all over the world, pupils should become acquainted with this system of measures. At the fourth grade level, the pupil is introduced to the centimeter only.

We have treated the inch, foot, yard, and mile as "standard" units for linear measure, in contrast to units of arbitrary size which may be used when communication is not important. Actually, the one standard unit for linear measure is the meter, and the correct sizes of such other units as the centimeter, inch, foot, and yard are specified with reference to the meter. Various methods for maintaining a model of the standard meter have been used by the Bureau of Standards. For many years the model was a platinum bar, kept under carefully controlled atmospheric conditions. The meter is now defined as having length which is 1,650,763.73 times the wave length of orange light from krypton 86. This standard for the meter is preferred because it can be reproduced in any good scientific laboratory and provides a more precise model than the platinum bar.

Scales and Precision

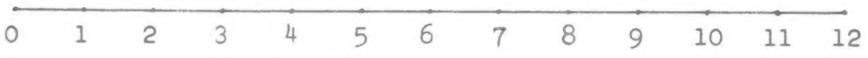
Once a unit is agreed upon, the creation of a scale greatly simplifies measurement. A scale can be made with a non-standard unit or with a standard unit. When we use a standard unit segment to mark off a scale on a straightedge, we are making a ruler. If we use the inch as the unit in making a ruler, we have a measuring device which is designed to give us readings to the nearest inch.

Because a line segment is continuous, any measurement of its length, made with a ruler, is, at best, approximate. When a segment is to be measured, a scale based on the unit appropriate to the purpose of the measurement is selected. The unit is the segment with endpoints at two consecutive scale divisions of the ruler. The scale is placed on the segment with the zero-point of the scale on one endpoint of the segment. The number which corresponds to the division point of the scale nearest the other endpoint of the segment is the measure of the segment. Thus, every measurement is made to the nearest unit. If the inch is the unit of measure for our ruler, then we have a situation in which two line segments, clearly not the same length, do have the same measure, to the nearest inch.



The measure, in inches, of $\overline{AB}$ to the nearest inch is 2. We write this, m $\overline{AB} = 2$. The measure, in inches, of $\overline{CD}$ to the nearest inch is also 2; m $\overline{CD} = 2$.

We may make a more precise measurement if we use a smaller unit segment. It should be clear that if the unit is changed, the scale changes. Thus, if we decide to use the centimeter as our unit, the scale appears thus:



m $\overline{AB}$ (in centimeters) = 4
m $\overline{CD}$ (in centimeters) = 6

Now the measures indicate that there is a difference in the lengths of the two segments.

Sometimes it is more convenient to record a length of 31 inches as 2 feet 7 inches. Whenever a length is recorded using more than one unit, it is understood that the precision of the measure is indicated by the smallest unit named. A length of the measure is indicated by the smallest unit named. A length of 4 yd. 2 ft. 3 in. is precise to the nearest inch. It is closer to 4 yd. 2 ft. 3 in. than it is to either 4 yd. 2 ft. 2 in. or 4 yd. 2 ft. 4 in. A length of 4 yd. 2 ft. is interpreted to mean a length closer to 4 yd. 2 ft. than to 4 yd. 1 ft. or 4 yd. 3 ft. However, if this segment were measured *to the nearest inch* we would have to indicate this by 4 yd 2 ft. 0 in. or 4 yd. 2 ft. (to the nearest inch). The difference in the precision of these measurements is marked. When the measurement is made to the nearest foot, the interval within which the length may vary is *one foot*; when the measurement is made to the nearest inch, the interval within which the length may vary is *one inch*.

An application of linear measure is the calculation of the perimeter of a polygon. As an introduction to perimeter, the pupil should be made aware of the fact that *length* is a property of a curve, which is *unchanged* if a representation of the curve is moved or if it bent and changed in shape. Thus a wire representation of any curve may be straightened out to represent a line segment and its length may then be measured with a ruler. The presentation in this unit stresses the concept of perimeter of a polygon as the length of the polygon. If the lengths of the sides of a polygon are known, we may find the perimeter readily by adding the measures of the segments, provided the same unit is used. The plausibility of this procedure arises from the definition of perimeter as a length, and the nature of length as unchanging when the curve is reshaped.

CONCEPT

The unit for measuring must be of the same nature as the thing to be measured: a line segment as a unit for measuring line segments, an angle as a unit for measuring angles, etc. For convenience in communication, standard units (foot, meter, degree, square foot, square meter, etc.) are used.

The measure of a geometric object (line segment, angle, plane region, solid region) in terms of a unit is the number (not necessarily a whole number) of times the unit will fit into the object.

Measurements yield underestimates and overestimates of measures in terms of whole numbers of units. In the case of line segments and angles they also yield approximations to the nearest whole number of units.

Segments and regions can be thought of as mathematical models of physical objects. Physical terms are used to describe the physical objects and the physical terms are also used in discussing mathematical models. This is acceptable provided the correct mathematical interpretation of the physical terms is understood.

A curve in space may have length.

Some measures of a figure may be calculated from other measures of that figure.

Disjoint segments (several separate pieces) may also have the property of length.

LENGTH

A line segment is a set of points consisting of two different points A and B and all points between A and B on the line containing A and B. Sometimes we say "segment" when it is clear that we mean "line segment".

We use a line segment as a unit for measuring line segments.

We use the word "meter" to name the segment which is accepted as the standard unit for linear measurement. We use "inch," "foot," and "yard" to name certain other units which are defined with relation to the standard unit.

The measure of a line segment in terms of a unit is the number (not necessarily a whole number) of times the unit will fit into the line segment. The unit segments may have common endpoints but must not overlap.

In measuring a line segment, as the unit becomes smaller, the interval within which the approximate length may vary, decreases in size. The precision of a measurement depends upon the size of this interval. The smaller the unit, the smaller the interval and the more precise the measurement.

The length of a line segment in terms of a unit consists of (1) the measure of this segment in terms of this unit together with (2) the unit used. Example: The length of this segment is 5 inches; its measure (in inches) is the number 5.

Many of the familiar curves in a plane or in space also have length. We can bend a wire to the shape of the curve and then straighten the wire to represent a segment.

We can calculate the perimeter of a triangle or other polygon. If the measures of the sides of a triangle (where the unit of measurement is the inch) are 4, 5, and 6, then the perimeter of the triangle is measured by the number 4 + 5 + 6 or 15. We say that the perimeter of the triangle is 15 inches.

A figure consisting of several segments that do not touch may have length. The measure of the figure in terms of a given unit is the sum of the measures of the separate segments in terms of that unit.

TEACHING THE UNIT

Each section of this unit is divided into activities of an exploratory nature and Exercises. In most sections, a generalization will be reached with the completion of the activities. However, in some cases, demonstrations or additional teacher-directed activities may be necessary in order to reach the desired generalization.

The pupils should work independently on the Exercises. Since the Exercises serve not merely for maintenance and drill, but also are sometimes developmental in character, it is suggested that class discussion of the Exercises follow their completion by the pupils.

You may want to have the pupils study something on the history of measure. There are several good reference books and commercial charts suitable for such work.

COMPARING SIZES

Objective: To develop the following understandings and skills:

1. The size of a collection of separate (discrete) objects can be found by counting the objects.
2. The size of a continuous object, such as a line segment, can not be found by counting only.
3. The sizes of two collections of discrete objects may be compared without counting, by a "matching" process.
4. The sizes of two line segments may be compared by laying a copy of one on the other.
5. A model of a curve may be "straightened" without changing its length.
6. A compass is a useful device for comparing sizes of segments.

Materials Needed:

Teacher:	Collections of small objects such as jacks, marbles, pencils, erasers, books, string, chalkboard compass, straightedge
Pupils:	String, compass, crayons

Vocabulary: Line segment, curve

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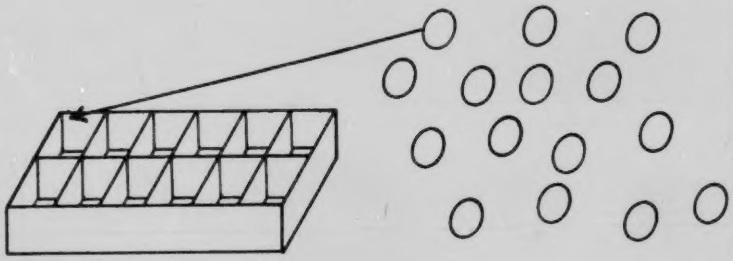
Questions About Size

This exploration serves as an introduction to the comparison of sizes.

Comparing Sizes Without Counting

After the pupils complete the first four exploratory exercises on page 461-2 in the pupils' book, activities similar to the one below should help give them more insight on comparing sizes without counting. Pupils should draw representations on their papers and match objects by drawing a line to match one member of a set with one member of the other set. They should be led to the generalization that one set is larger than the other set if, after matching, it then has unmatched members.

Below is an egg box and a set of eggs. The line matches one egg with one place in the box. Represent the box and the eggs on your paper. Match each egg with a place in the box, if possible.



Are there any spaces without eggs?
Are there any eggs without spaces?
Which set is larger?

Using a Compass to Compare Segments

The use of the compass to compare segments is reviewed in the Exploration of the section on Using a Compass to Measure Line Segments.

Chapter 9

LINEAR MEASUREMENT

COMPARING SIZES

Questions About Size

Exploration

How many of the questions you ask and other people ask begin, "How much?" "How many?" "How far?"

We say: "How many pupils are there in your class?"

"How long is the hall outside your classroom?"

"How far is it to New York?"

To answer these questions, we use numbers.

There are 32 pupils in the class.

The hall is 320 feet long.

It is 750 miles to New York.

Answers to some of these questions can be found by counting. Other answers can not be found just by counting. Why? (Because the sets involved contain more points than can be counted.)

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Which of these can be found just by counting? (1, 3, 6, 8)

1. The size of your class
2. The height of the tallest boy in your class
3. The size of your family
4. The size of your classroom
5. The length of a book shelf
6. The size of a rock collection
7. The size of the smallest rock in the collection
8. The population of your town
9. How hard the wind is blowing
10. The size of a bicycle wheel

Think of a rock collection. What can you tell about the rock collection by counting? Is there something you cannot tell about the size of the rock collection by counting?

You can tell the size of a rock collection by counting, because each rock is a separate thing. A rock may be large or small, but it is one rock.

You cannot tell the size of any single rock by just counting. It is one rock, but it may be large or small.

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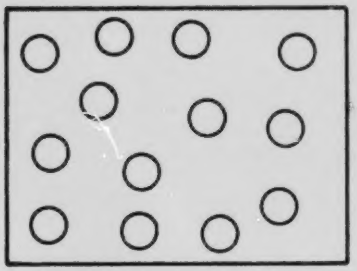
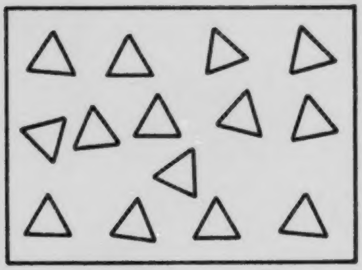
COMPARING SIZES WITHOUT COUNTING

Exploration





Without counting, how can you tell which set has more members? (by matching)

1. A scout brings a bag of candy bars to a den meeting. What is an easy way to tell, without counting, whether there are more scouts or more candy bars? (Give each scout one candy bar.)

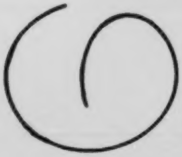
2. In a school storeroom there is a supply of desks and a supply of chairs. How can you tell which supply is larger? (Put one chair with each desk.)

3. How can you tell how the attendance at a movie compares with the seating capacity? (See whether all seats are occupied and whether someone has no seat.)

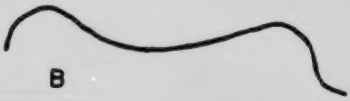
4. How can you tell whether there are more hot dogs or more buns? (If possible, put exactly one hot dog in each bun.)

You can compare the sizes of single objects whose size can not be found by just counting.

5. Here are pictures of two pieces of rope.



A



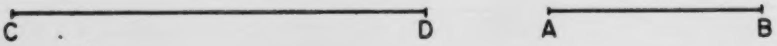
B

Which rope do you think is longer, A or B?

If you had the ropes instead of the pictures, how could you tell which is longer?

To compare curves like those in the pictures, lay a string along each curve. Then straighten out the strings. What geometric figures do the stretched-out strings represent? (line segments)

6. Suppose you wish to compare $\overline{AB}$ and $\overline{CD}$.



To prove to someone that $\overline{CD}$ is longer than $\overline{AB}$, stretch a string on $\overline{AB}$. With your fingers, hold the points of the string that fall on A and B. Move the string and place it on $\overline{CD}$ with one endpoint on C. Is the other endpoint on D? Since it is between C and D, $\overline{CD}$ is longer than $\overline{AB}$. How is comparing segments, to see which is the longer, something like matching groups of separate objects to compare the sizes of the groups without counting? (In both cases, we compare directly instead of using numbers in estimating the sizes involved.)

7. There are many things which are enough like our idea of a line segment so that we can think of them as line segments; for example,

a stretched string,
some pencil marks on paper,
the edge of a table,
a pencil.

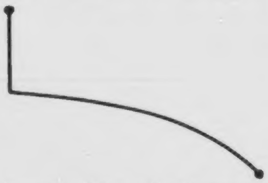
Name several other things which we might think of as line segments.

8. Remember that between any two points in space there is just *one* line segment. If we have any two objects which we *think of* as *points*, we can also think of them as endpoints of a line segment. This is what we mean when we say:

> the distance between the two tips of a compass,
>
> the distance between the earth and a star,
>
> the distance from home plate to second base,
>
> the height at which an airplane flies.

Name several other ways we think of line segments by thinking of objects as their endpoints.

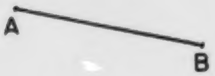
9. The way you go to school is probably not much like a line segment. A picture of it might look like this:

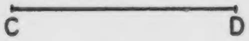


We can still talk about the distance you travel in going to school. Your path can be thought of as a *curve*, but not as a *line segment*. What does distance or length mean for curves? We think of the curve as represented by a piece of wire or string. Then we imagine straightening out the wire or string to represent a line segment.

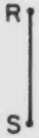
Exercise Set 1

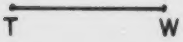
Use strings to compare the line segments and other curves. Copy each sentence below the figures and write "longer" or "shorter" in the space.



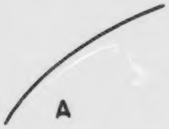


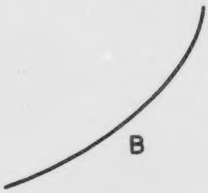
$\overline{AB}$ is (shorter) than $\overline{CD}$.



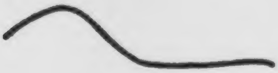


$\overline{RS}$ is (shorter) than $\overline{TW}$.

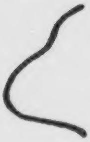




Curve A is (shorter) than curve B.

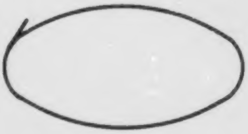


C

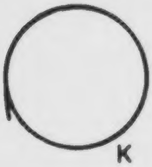


D

Curve C is (longer) than curve D.

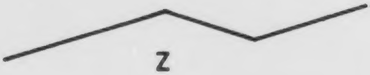


M

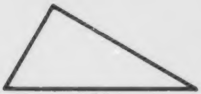


K

Curve M is (longer) than curve K.



Z



N

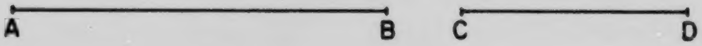
Curve Z is (shorter) than curve N.

P467

USING A COMPASS TO COMPARE SEGMENTS

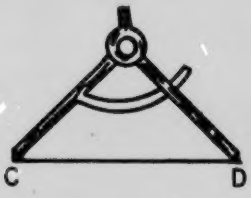
Exploration

Is $\overline{AB}$ longer than $\overline{CD}$?

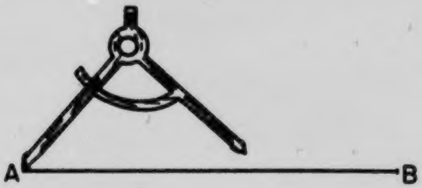


Recall how you have used a compass to compare segments.

Place the points of your compass on C and D.

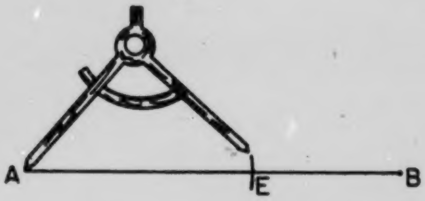


Without changing your compass, place the sharp point on A.



Draw a small part of a circle which intersects $\overline{AB}$.

Label the intersection E.



E is between A and B, so $\overline{AB}$ is longer than $\overline{AE}$.

$\overline{AE}$ has the same length as $\overline{CD}$, so $\overline{AB}$ is longer than $\overline{CD}$.

Exercise Set 2

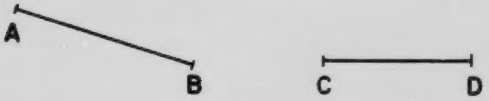
Trace the following figures on your paper.

Use your compass or string to help you compare segments.

Copy and answer the question for each exercise.

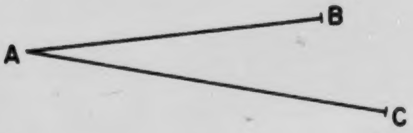
Color the unmatched part of the longer segment.

1.



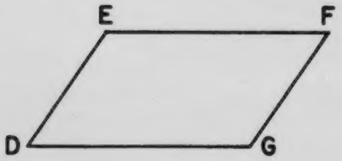
Which is longer, $\overline{AB}$ or $\overline{CD}$? $(\overline{AB})$

2.



Which is longer, $\overline{AB}$ or $\overline{AC}$? $(\overline{AC})$

3.



Which is longer, $\overline{DE}$ or $\overline{EF}$? $(\overline{EF})$

Measuring a Segment

You may wish to use some measuring instrument other than the pencil or hand-spans. Toothpicks, chalkboard erasers, or any other set of objects all of the same length would do. In addition to the activities suggested in the Exploration, give the pupils other experiences in measuring objects in the classroom using non-standard unit segments. Use as measuring instruments such things as these: a piece of string with two knots about ten inches apart; a pupil's foot; a walking step; the edge of a sheet of paper. You may wish to engage in more activities similar to the one given in the Exploration on Measuring to the Nearest Unit of the pupil text before assigning Exercise Set 5. Discuss the results obtained to emphasize:

(a) The unit of measure for a line segment is a line segment.

(b) The measure is a number (e.g., the measure of $\overline{CD}$ may be 7).

(c) The length of a segment includes both the measure and the unit of measure. (e.g., 3 inches, 7 feet, etc.)

(d) If a segment is measured with different units, the larger unit of measure is associated with the smaller measure. For example, the measure of the desk edge in pencil units will be larger than its measure in hand-span units, if the hand-span is longer than the pencil.

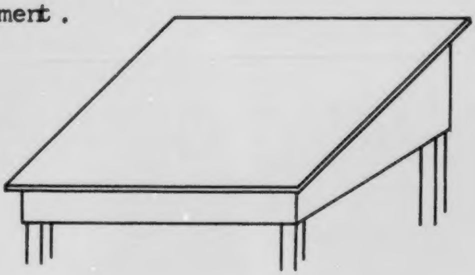
(e) The measuring instrument must be used with care, so that successive unit segments intersect in their endpoints, and only in their endpoints (i.e., they have one and only one point in common.

MEASURING SEGMENTS

Measuring a Segment

Exploration

You know that you cannot tell the size of a segment by just looking at it and counting. You can tell how long it is by comparing it with some other segment.



How can you tell how long your desk is? (by comparing it to some other segment.)

What geometric figure does the edge of your desk represent? (a line segment)

1. Take your pencil.

   Lay it along the edge of your desk, with one end on the corner.

   Put your finger on the desk at the other end of the pencil.

   Lay the pencil down again from that point.

   How many pencils long is the edge of your desk? (Answers will vary)

   Give your answer to the nearest whole number.

2. People often use their hands to tell how long a segment is. Spread your right hand so the fingers are as far apart as possible. Place your right hand with your thumb on the left corner of your desk. See how many hand-spans long your desk is. (Answers will vary.)

The segment represented by your pencil or your hand-span is called your unit of measure.

The number of pencil-lengths or hand-spans it took to cover the edge of your desk is the measure of your desk. A measure of your desk may be 7.

To name a length, we use both the measure and the unit of measure. A length of your desk may be 5 hand-spans.

Exercise Set 3

Copy and complete each of the sentences.

1. Our family drinks 3 quarts of milk each day.
   a. The unit of measure is (quart).
   b. The measure is (3).
   c. The amount of milk is (3 quart).

2. My dog weighs 18 pounds.
   a. The unit of measure is (pound).
   b. The measure is (18).
   c. Its weight is (18 pounds).

3. My desk is 9 chalk-pieces long.
   a. Its length is (9 chalk-pieces).
   b. Its measure is (9).
   c. The unit of measure is (chalk-piece).

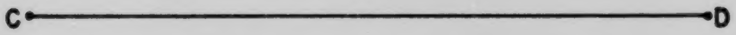
In Exercises 4-9 use the words length, measure, or unit of measure, so that each sentence will make sense.

4. It takes 6 chalk-pieces to cover the edge of my desk.
   Its (measure) is 6.
5. My desk is 4 pencils long.
   Its (length) is 4 pencils.
6. My desk is 4 hand-spans long.
   The (unit of measure) is the hand-span.
7. The (measure) in hand-spans is 5.
8. The (unit of measure) I used is a segment represented by one pencil.
9. The edge of my desk has a (length) of 4 pencils.

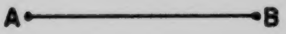
USING A COMPASS TO MEASURE LINE SEGMENTS

Exploration

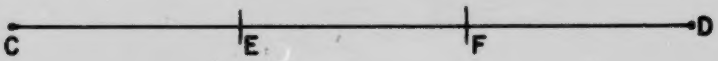
We want to find the measure of $\overline{CD}$.
We use our compass to help us
measure a line segment



Our unit of measure is $\overline{AB}$.
Here is $\overline{AB}$.



We lay off the unit $\overline{AB}$ on $\overline{CD}$.
We lay off $\overline{AB}$ on $\overline{CD}$ three times.
We label the intersections E and F.
See the picture below.



We say the measure of $\overline{CD}$ is 3.
We write: m $\overline{CD} = 3$.

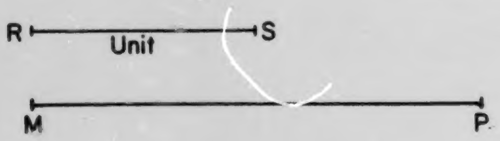
What is the measure of $\overline{CE}$? (m $\overline{CE} = 1$)
What is the measure of $\overline{CF}$? (m $\overline{CF} = 2$)
What is the measure of $\overline{ED}$? (m $\overline{ED} = 2$)

Exercise Set 4

Copy each of the segments in exercise 1 and 2.

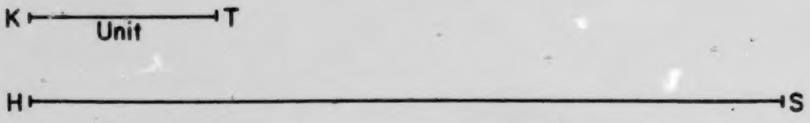
Find the measure of each segment.

1.



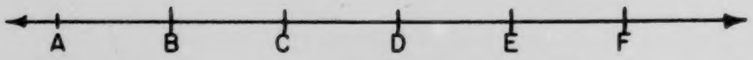
m $\overline{MP}$ = (2)

2.



m $\overline{HS}$ = (4)

3. Tell the measures of the segments in the figure.



m $\overline{AE}$ = (4)

m $\overline{BE}$ = (3)

m $\overline{EF}$ = (4)

m $\overline{AD}$ = (3)

m $\overline{CF}$ = (3)

m $\overline{AF}$ = (5)

4. Write the name of any segment which has the measure given. Use the unit and figure for Exercise 3.

m $\overline{AC}$ = 2

(m $\overline{BD}$ = 2
m $\overline{CE}$ = 2
m $\overline{DF}$ = 2)

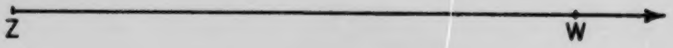
m $\overline{AB}$ = 1

(m $\overline{BC}$ = 1
m $\overline{CD}$ = 1
m $\overline{DE}$ = 1
m $\overline{EF}$ = 1)

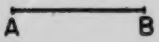
MEASURING TO THE NEAREST UNIT

Exploration

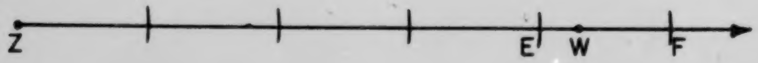
We want to find the measure of $\overline{ZW}$.



We use $\overline{AB}$ as the unit of measure.



This picture shows how we can find the measure of $\overline{ZW}$, using $\overline{AB}$ as unit.



How many times can you lay off the unit on $\overline{ZW}$?

m $\overline{ZE}$ = 4. Is m $\overline{ZW}$ larger than 4? (yes) Why? (because E is between Z and W)

m ZF = 5. Is m $\overline{ZW}$ smaller than 5? (no) Why? (because W is between Z and F)

The measure of $\overline{ZW}$ is between 4 and 5 units.

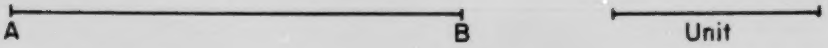
Since W is nearer to E than F, we say

m $\overline{ZW}$ = 4, to the nearest unit.

Exercise Set 5

Trace the figures in each of the following exercises. Find the measures of the segments to the nearest unit.

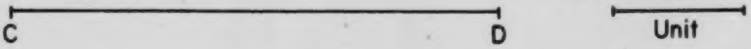
1.



The length of $\overline{AB}$ is greater than (2) units but less than (3) units.

m $\overline{AB}$ = (2) (to the nearest unit)

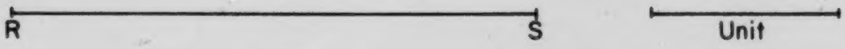
2.



The length of $\overline{CD}$ is greater than (3) units but less than (4) units.

m $\overline{CD}$ = (4) (to the nearest unit)

3.



The length of $\overline{RS}$ is greater than (2) units but less than (3) units.

m $\overline{RS}$ = (3) (to the nearest unit)

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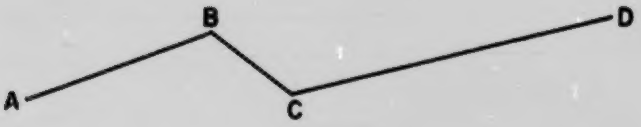
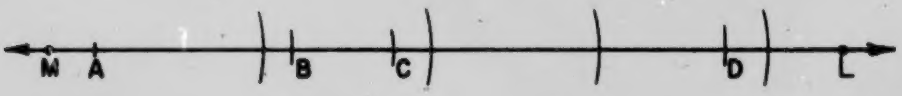
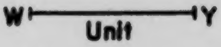
BRAINTWISTER:

On $\overleftrightarrow{ML}$ draw a segment whose length is the length of the curve ABCD.





Now find the measure of curve ABCD. Use $\overline{WY}$ as unit.



The length of curve ABCD is greater than (4) units but less than (5) units.

m ABCD = (4) (to the nearest unit)

USING STANDARD UNITS OF LENGTH

Objective: To develop the following understandings and skills:

1. Units of measure of standard size came into use for purposes of communication. Their size is fixed by law.

2. Many units of measure which are now of standard size developed from non-standard units.

3. Two unit segments of standard size are the inch in the British-American System, and the centimeter in the Metric System.

Materials Needed:

Teacher: Chalkboard compass, straightedge

Pupils: Compass, scissors, straightedge

Vocabulary: inch, centimeter

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Children may be interested in the ways in which the local government controls the measuring instruments used by merchants. For interesting reading on the development of measurement, see

Hogben, Lancelot Thomas, Wonderful World of Mathematics, 1955 (Garden City) Doubleday.

Newman, James R., ed., World of Mathematics, 1956 Simon and Schuster.

Interesting reports on topics relating to measurement may be made by pupils, using encyclopedias and other reference sources.

Following the section on Using Standard Units of Length in the pupil's book, an oral exercise in which the pupils supply the name of a standard unit which would make sentences sensible would be in order. The following sentences are samples of such an exercise.

The school nurse says I weigh 65 (pounds).

My mother buys 3 (gallons, quarts, or pints) of milk a day.

This was a warm day. The thermometer read 80 (degrees).

My father stopped at a filling station and bought 10 (gallons) of gas.

We drove to a town 80 (miles) away to see my aunt. It took us 2 (hours) to get there.

In the pupil's books, the compass marks do not show on the segments represented in Exercises 1-8 in Exercise Set 6.

In this section the pupils make and use an inch and a centimeter scale.

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USING STANDARD UNITS OF LENGTH

Exploration

Suppose a team of boys from your school were going to play a game of baseball with a team from another school. If the other team brought a baseball so large and heavy you could hardly lift it, what would you say? You would probably say, "We will not play with that baseball. It is not the standard size and weight." What does that mean? Can you find out the standard size of a baseball?

You have been using units of measure which were not of "standard" size. Now we shall use standard units, which are used by a great many people and which always mean the same amount. The size of a standard unit is set by law. Your encyclopedia contains information about the National Bureau of Standards in Washington, D. C.

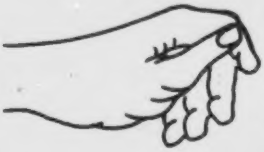
In which of the following sentences are standard units used? (3)

1. He is as strong as an ox.
2. Put in a pinch of salt.
3. We get $\frac{1}{2}$ pint of milk for lunch.
4. The corn is knee high.
5. I used to live a day's journey from here.

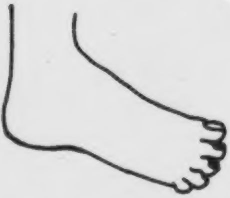
You are familiar with many standard units. Name some of these units. (pound, foot, yard, etc.)

Primitive people had little need for standard units. If the caveman liked the size of his neighbor's spear, he could borrow the spear and copy its length. Or, he could think, "When the spear is held with one end on the ground, the other end reaches my shoulder." Then, he could make a spear with that same length.

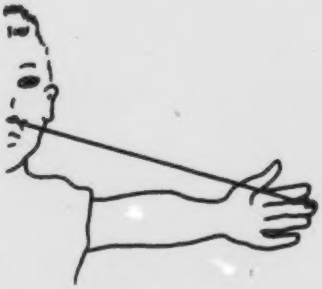
Many of the standard units which we use came from units which were not standard. These were used by people long ago. Many of them came from using parts of the body.



An inch came from the use of a part of a finger as a unit of length. Can you find a part of your finger which is about an inch long?



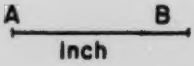
A foot came from the length of a person's foot. Is your foot shorter or longer than a foot ruler?



A yard was at one time thought of as the distance from the tip of a person's nose to the tip of his middle finger, when his arm was held straight out from the shoulder. Is the distance from your nose to the tip of your finger as long as a yardstick?

Here is a model of a standard unit.

You have often used this unit.

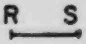


It is the inch.

Name some objects you measure in inches.

Here is a model of another standard unit.

This unit may be new to you.



It is the centimeter.

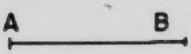
If you lived in France, or in many other countries, you would use this unit segment instead of the inch. Scientists in all countries use this unit.

Exercise Set 6

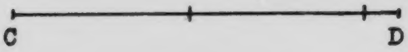
Trace each figure in exercises 1-8 on your paper. Use your compass or string to find the measures of the segments.

Copy and complete the statement for each exercise.

For exercises 1-4, use $\overline{AB}$ as the unit segment. Give your answer to the nearest inch.

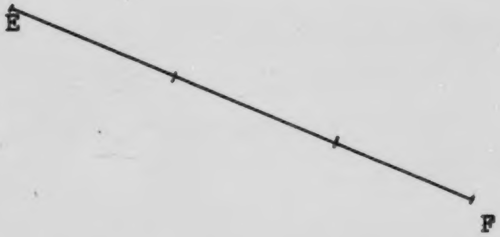


1.



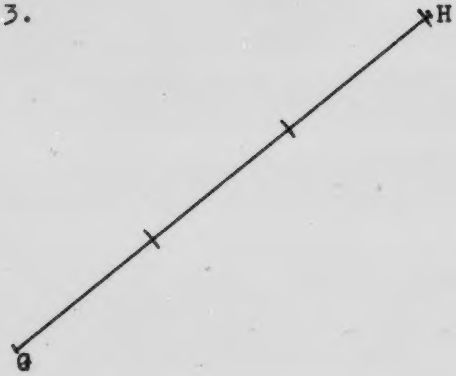
m $\overline{CD}$ = (2), in inches

2.



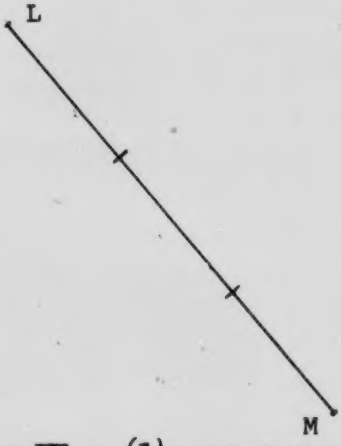
m $\overline{EF}$ = (3), in inches

3.



m $\overline{GH}$ = (3), in inches

4.

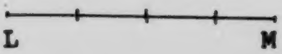


m $\overline{LM}$ = (3), in inches

For exercises 5-8, use $\overline{RS}$ as the unit segment. R —— S

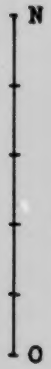
Give your answer to the nearest centimeter.

5.



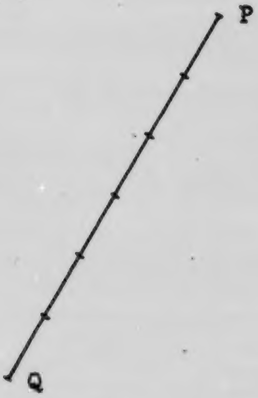
m $\overline{LM}$ = (4),
in centimeters.

6.



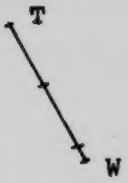
m $\overline{NO}$ = (5),
in centimeters.

7.



m $\overline{PQ}$ = (6),
in centimeters.

8.



m $\overline{TW}$ = (2),
in centimeters.

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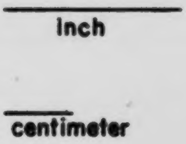
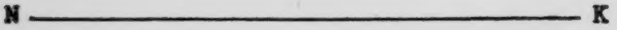
9. Make two copies of $\overline{NK}$.

Find the measure of $\overline{NK}$ in inches.

Then find its measure in centimeters.

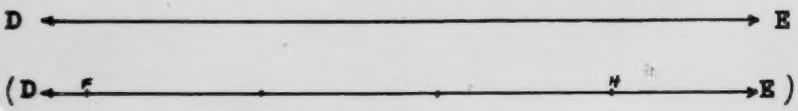




m $\overline{NK}$ (in inches) = (3)

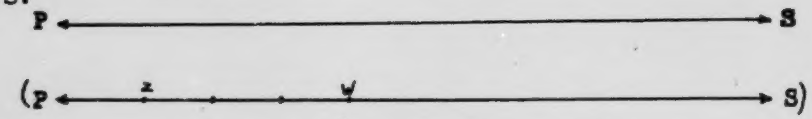
m $\overline{NK}$ (in centimeters) = (8)

10. Draw $\overleftrightarrow{DE}$.



On $\overleftrightarrow{DE}$ draw a segment whose measure, in inches, is 3. Label it $\overline{FH}$.

Draw $\overleftrightarrow{PS}$.



On $\overleftrightarrow{PS}$, draw a segment whose measure in centimeters is 3. Label it $\overline{ZW}$.

m $\overline{FH}$ (in inches) = 3

m $\overline{ZW}$ (in centimeters) = 3

Is $\overline{FH}$ congruent to $\overline{ZW}$? (No)

SCALES OF MEASURE

Objective: To develop the following understandings and skills:

1. A linear scale is a device for convenience in measuring segments.

2. A linear scale is constructed by laying off unit segments consecutively on a ray. Endpoints of consecutive segments (called scale division points) are assigned consecutive whole numbers, with 0 assigned to the endpoint of the ray.

3. A compass and linear scale may be used to measure a given segment.

4. When a linear scale is used to measure a segment, the measure is determined by the number assigned to the scale division point nearest the endpoint of the segment.

Materials Needed:

Teacher: Straightedge, chalkboard compass
Pupils: Compass

Vocabulary: Scale, linear scale, $\overrightarrow{r}$ (ray r), inch scale, scale division.

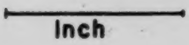
SCALES OF MEASURE

Making Inch and Centimeter Scales

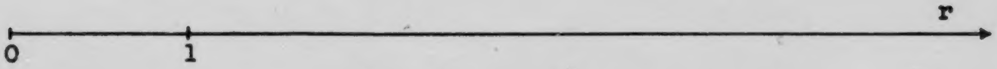
Exploration

When you are measuring segments, it is convenient to have a scale.

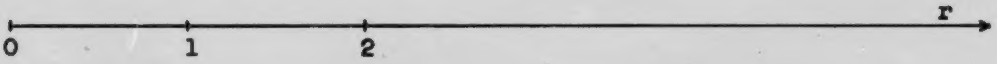
Follow these directions to make an inch scale on ray r. (Sometimes we name a ray by a single letter as $\overrightarrow{r}$.)



1. Write zero below the endpoint of $\overrightarrow{r}$. Then use your compass to copy the inch unit segment, beginning at 0. Label the other endpoint 1.

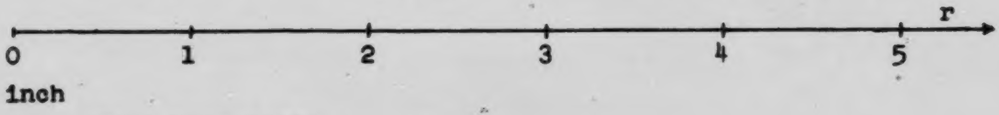


2. Copy the inch unit segment again, beginning at 1. Label the other endpoint 2.



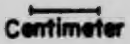
3. Continue copying the unit segment until you have copied it five times. Label the endpoints of the unit segments. Write "Inch" below 0.

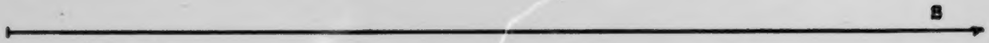
   Does your drawing now look like this one?



Save your scale.

4. Trace ray s. On ray s make a centimeter scale.

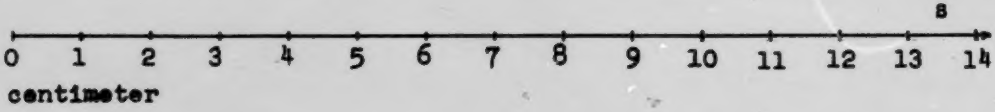




5. What is the largest number on your centimeter scale? (13 or 14)

What is the largest number on your inch scale? (5)

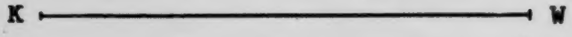
Does your centimeter scale look like this one?



Save your centimeter scale, too.

6. You can use your scales to find the measure of a segment in inches and in centimeters.

Copy and find the measure of $\overline{KW}$, in inches.



Place your compass with the points on K and W. Without changing your compass, place the sharp point on the zero-point of the inch scale and the pencil point on the ray. What point on the scale is nearest the pencil point? (3) What is its number? (3) What is the measure of $\overline{KW}$ in inches? (3)

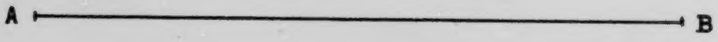
7. Find the measure of $\overline{KW}$ in centimeters. (7) Use your compass and scale.

Exercise Set 7

Using your inch scale and your centimeter scale, find the measures of these segments.

Find the measures in inches and in centimeters.

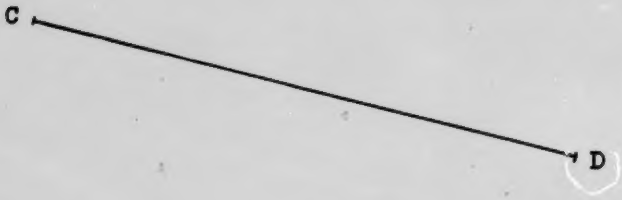
1.



m $\overline{AB}$ (in inches) = (4)

m $\overline{AB}$ (in centimeters) = (9)

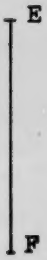
2.



m $\overline{CD}$ (in inches) = (3)

m $\overline{CD}$ (in centimeters) = (8)

3.



m $\overline{EF}$ (in inches) = (1)

m EF (in centimeters) = (3)

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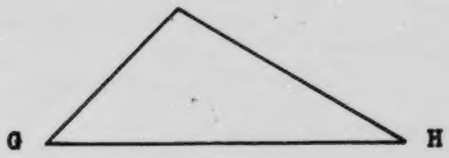
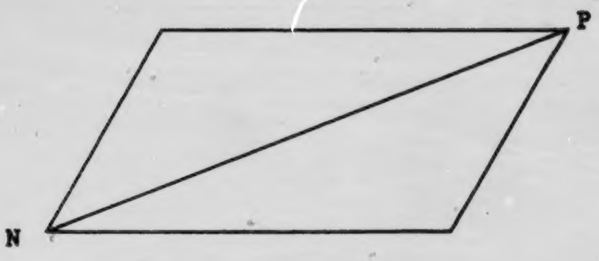
In these figures, the endpoints of one segment are named. Find the measure of the segment, in centimeters.



m $\overline{GH}$ (in centimeters) = (5)

5.



m $\overline{NP}$ (in centimeters) = (8)

USING LINEAR SCALES OF MEASURE

Objective: To develop the following understandings and skills:

1. A straightedge marked with a linear scale based on a unit segment of standard size is a ruler.
2. To measure a segment with a ruler, place the ruler along the segment with one scale division point of the ruler on one endpoint of the segment.
3. If the 0-point of the scale is on one endpoint of the segment, the measure of the segment is the number of the scale division point nearest the other endpoint. If the 0-point is not used, the measure is the difference between the numbers at the endpoints of the segment. Note that sometimes we place a ruler in different ways because the end of the ruler is worn away.

Materials Needed:

Teacher: Straightedge, compass

Pupils: Straightedge, compass, cardboard ruler

Vocabulary: Ruler

> You may wish to suggest that, for careful measurement, it is helpful to use a compass and ruler, in the same way as the compass and linear scale were used in the previous section.
>
> The Inch Scale and the Centimeter Scale
>
> In the previous exploration a scale and a compass were used. This exploration introduces the use of the scale alone. Many children will be familiar with the inch scale but they probably have had little opportunity to work with the centimeter scales.

THE INCH SCALE AND THE CENTIMETER SCALE

Exploration

You have made an inch scale and a centimeter scale.

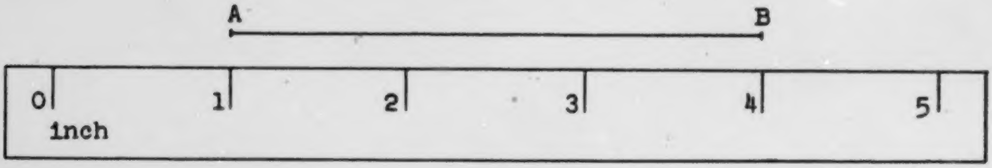
You have used these scales to find measures of line segments.

You have found measures of line segments in other ways, too.

Now we will use only the scales of measure.

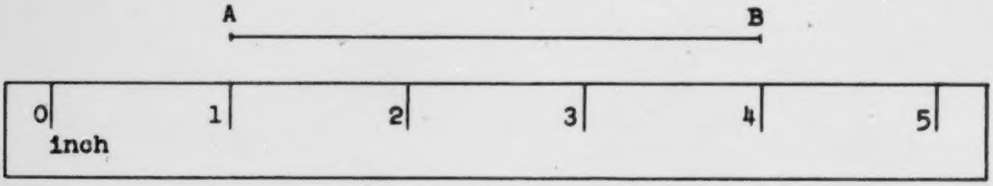
We find the length of $\overline{AB}$.

1. We can place our scale along $\overline{AB}$ like this:



2. Here is another way of placing our scale.

Why would you place the ruler in this way? (A ruler may be worn away at the end so that the scale is not clearly shown. This could be so if the zero mark is exactly on the end, as it is on many commercial rulers.)



3. Are there other ways to place the ruler to find the length of $\overline{AB}$? (Yes, by placing A at 2, or at any scale division point.)

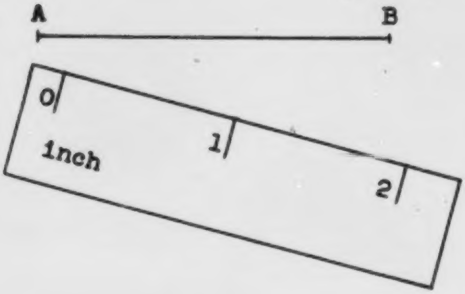
The length of $\overline{AB}$ is 3 inches.

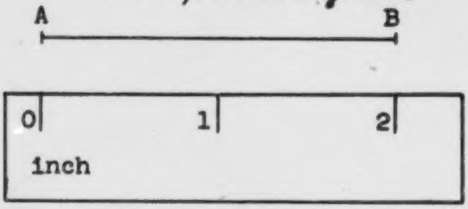
We write $m\,\overline{AB} = 3$.

We read: The measure of $\underline{AB}$ is 3.

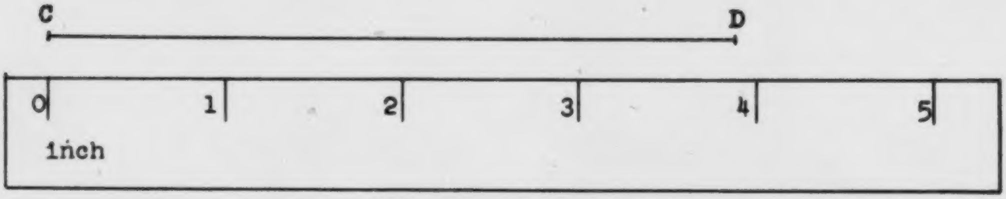
When you used the scale to find the measure of $\overline{AB}$ you used it as a ruler. A ruler is a straightedge with a scale on it.

4. Which of these drawings shows the correct way to place a ruler to measure a segment? (the second) Why? (The segments to be compared need to be put side by side.)





5. Find the length of $\overline{CD}$, to the nearest inch.



Place the scale along $\overline{CD}$ as in the picture.

What point on the scale is below C? (0)

Point D lies between what two points on the scale? (3 and 4)

Is D closer to 3 or 4 on the scale? (4)

What is the length of $\overline{CD}$, to the nearest inch? (4 inches)

Give two other ways to use the scale to measure $\overline{CD}$.
(Place 1 below C or 2 below C.)

6. Here are pictures of some line segments.

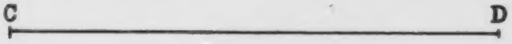
Find the length of each segment to the nearest inch.

Then, find the length of each segment to the nearest centimeter.

The abbreviation for "centimeter" is "cm."

Write your answers for each as has been done in the first one.

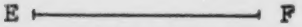
a)



Length of $\overline{CD}$ is 3 in., to the nearest in.

Length of $\overline{CD}$ is 7 cm., to the nearest cm.

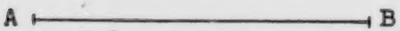
b)



(Length of $\overline{EF}$ is 1 in., to the nearest inch

Length of $\overline{EF}$ is 3 cm., to the nearest cm.)

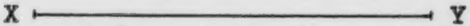
c)



(Length of $\overline{AB}$ is 2 in., to the nearest inch.

Length of $\overline{AB}$ is 5 cm., to the nearest cm.)

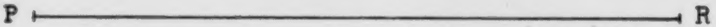
d)



(Length of $\overline{XY}$ is 2 in., to the nearest inch.

Length of $\overline{XY}$ is 6 cm., to the nearest cm.)

e)



(Length of $\overline{PR}$ is 4 in., to the nearest inch.

Length of $\overline{PR}$ is 9 cm., to the nearest cm.)

7. Draw a segment 8 cm. in length.

8. Draw a segment 2 in. in length.

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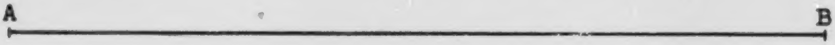
Exercise Set 8

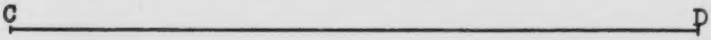
1. Find the length, to the nearest inch, of one of your pencils. (Answers will vary.)

2. Find the length of the same pencil in centimeters. (Answers will vary.)

3. What is the length of $\overline{AB}$, in inches? (5 in.) What is its length in centimeters? (12 cm)



4. $\overline{CD}$ is 4 inches long. What is its length in centimeters? (10 cm.)



5. Draw a segment that has a length of 5 inches.

6. Draw a segment whose measure, in inches, is 4.

7. Draw a segment whose measure, in centimeters, is 10.

8. Draw a segment that has a length of 14 centimeters.

9. Copy the sentence and write the words inches or centimeters in the blanks so as to make the sentence true.

A segment which is 5 (inches) in length is longer than a segment which is 5 (centimeters) in length.

OTHER STANDARD UNITS

Objective: To develop the following understandings and skills:

1. In the British-American System for linear measurement, 1 foot is equivalent to 12 inches, 1 yard is equivalent to 3 feet, and 1 mile is equivalent to 5280 feet.
2. For convenience, lengths may be expressed in more than one unit, e.g. 2 feet 3 inches.
3. The scale used for measuring a segment is shown by the smallest unit named in the length.
4. To find the combined length of two segments, or to find how much longer one segment is than another, measures which relate to the same unit of measure may be added or subtracted.

Materials Needed:

Teacher: Foot ruler, yardstick

Pupils: Cardboard ruler, foot ruler

> You will need to plan the activities of this section carefully. Individual or group work at the chalkboard may be effective. You may prefer to let groups work together on large pieces of wrapping paper. Each child should participate and actually perform the measurement.

OTHER STANDARD UNITS

Exploration

The inch unit and the centimeter unit are too small to be used for measuring long distances. As you know, larger units are used for this purpose. Three which you know are named the foot (ft.), the yard (yd.), and the mile (mi.).

Recall what you know about these unit segments.

1. Draw a segment 1 foot long. Draw a segment 1 yard long.

2. Draw other segments 2 ft., 3 ft., and 4 ft. long.

3. Measure each of the segments in inches. List your results like this:

Length in feet	Length in inches
1 ft.	(12) inches
2 ft.	(24) inches
3 ft.	(36) inches
4 ft.	(48) inches

Which of these segments was one yard in length? (3 ft. or 36 in.)

4. Draw a line segment 18 inches long. Name it $\overline{AB}$.
Mark a point C on $\overline{AB}$ so that $\overline{AC}$ is 1 ft. long.
How long is $\overline{CB}$? (6 inches)
The length of $\overline{AB}$ is 1 ft. (6) in.

5. Draw a line segment 29 in. long. Name it $\overline{AB}$.

Find a point C on $\overline{AB}$ so that $\overline{AC}$ is 1 ft. long.

Then find a point D on $\overline{CB}$ so that $\overline{CD}$ is 1 ft. long.

How long is $\overline{AD}$? (2 ft. or 24 in.)

How long is $\overline{DB}$? (5 in.)

$\overline{AB}$ is 2 ft. 5 in. long.

6. Draw segments with these lengths in inches.

Then tell the length in feet and inches.

Segment	Length in inches	Length in feet and inches
$\overline{EF}$	15 in.	(1 ft. 3 in.)
$\overline{GH}$	20 in.	(1 ft. 8 in.)
$\overline{IJ}$	27 in.	(2 ft 3 in.)
$\overline{KL}$	30 in.	(2 ft 6 in.)
$\overline{MP}$	36 in.	(3 ft 0 in.)

7. On $\overleftrightarrow{m}$ mark off $\overline{AB}$ whose length is 1 foot. Then mark off $\overline{BC}$ whose length is 3 inches so that C is not on $\overline{AB}$. What is the length of $\overline{AC}$ in feet and inches? (1 ft. 3 in.) What is the length of $\overline{AC}$ in inches? (15 in.) (Note: C is on $\overleftrightarrow{AB}$ but not on $\overline{AB}$.)

8. On $\overleftrightarrow{k}$ mark off $\overline{RS}$ whose length is 2 feet. Then mark off $\overline{ST}$ whose length is 7 inches, so that T is not on $\overline{RS}$. What is the length of $\overline{RT}$ in feet and inches? (2 ft. 7 in.) What is the length of $\overline{RT}$ in inches? (31 in.)

9. Draw segments of the lengths given. Then tell the length in another way.

Segment	Length in feet and inches	Length in inches
$\overline{AB}$	2 ft. 1 in.	(25 in.)
$\overline{CD}$	(1 ft. 9 in.)	21 in.
$\overline{EF}$	1 ft. 7 in.	(19 in.)
$\overline{GH}$	(2 ft. 8 in.)	32 in.

BRAINTWISTERS

1. A mile is a distance of 5,280 feet. About how many steps would you take to walk a mile? (The answers will vary.)

2. If a car is traveling 60 miles an hour, how far will it travel in one minute? (1 mile)
   How long does it take you to walk a mile? (The answers will vary.)

3. Suppose someone ran a mile in 238 seconds. How many minutes and seconds did he take? (3 minutes and 58 seconds)

4. Suppose the four men in a mile relay event ran their "quarter" mile in the times of 46 seconds, 47 seconds, 47 seconds, and 45 seconds. What was the time for the relay team to run the mile? (185 seconds, or 3 minutes and 5 seconds)

Exercise Set 9

1. If segments have these lengths, which is longer?

a) 10 inches or 1 foot (1 ft.)

b) 4 feet or 1 yard (4 ft.)

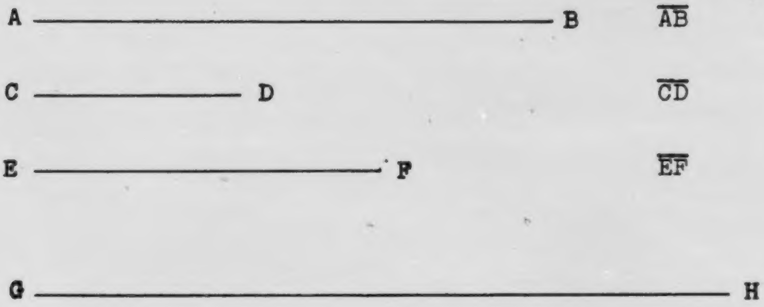
c) 1 inch or 1 centimeter (1 in.)

d) 1 foot or 10 centimeters (1 ft.)

e) 1 ft. 7 in. or 2 ft. (2 ft.)

f) 28 inches or 2 feet (28 in.)

2. Look at each segment and its measure, then write the unit.



Segment	Length
$\overline{AB}$	3 (in.)
$\overline{CD}$	3 (cm.)
$\overline{EF}$	5 (cm.)
$\overline{GH}$	4 (in.)

3. Complete:

	Length in feet and inches	Length in inches
$\overline{XY}$	3 ft. 1 in.	(37 in.)
$\overline{MN}$	(2 ft. 6 in.)	30 in.
$\overline{DE}$	1 ft. 10 in.	(22 in.)
$\overline{FG}$	(3 ft. 8 in.)	44 in.

4. Write the name of the unit which makes the sentence reasonable. Use centimeters, inches, feet, yards, or miles.

   a) The seat of the teacher's chair is 16 (inches) above the floor.

   b) Helen is in the fourth grade. Her height is 130 (centimeters).

   c) The lake is 4 (miles) long.

   d) The height of a tall man is 2 (yards).

   e) A crayon is about 10 (centimeters) long.

   f) It is 2 (miles) from Bob's house to the park.

5. If segments have these lengths, which is longer? How much longer?

		Which is Longer?	How much Longer?
a)	23 inches or 1 foot	(23 in.)	(11 in.)
b)	18 inches or 2 feet	(2 ft.)	(6 in.)
c)	4 feet or 1 yard	(4 ft.)	(1 ft.)
d)	1 ft. 8 in. or 16 in.	(1 ft 8 in.)	(4 in.)
e)	1 yd. 4 in. or 42 in.	(42 in.)	(2 in.)
f)	1 yd., 2 ft. or 7 ft.	(7 ft.)	(2 ft.)
g)	1 mile or 3,495 feet	(1 mile)	(1,785 ft.)

6. You know that 1 mile is 5,280 feet. How many yards is that? (1,760 yds.)

7. A Boy Scout can walk a mile in 12 minutes if he uses the Boy Scout pace. How many miles can he go in one hour? (5 miles)

P495

COMBINING LENGTHS

Learning to Combine Lengths

Exploration

1. The teacher made a record of the heights of children in his fourth grade class.

Pupil	Height
Bill	4 ft. 8
Betty	4 ft. 2
Jim	5 ft.
Helen	4 ft. 6

The children knew that he meant Bill's height was 4 feet and 8 inches. Betty's height was 4 feet and 2 inches. Jim's height was 5 feet and 0 inches. How many feet and inches is Helen's height? (4 ft. 6)

We often use more than one unit to tell sizes.

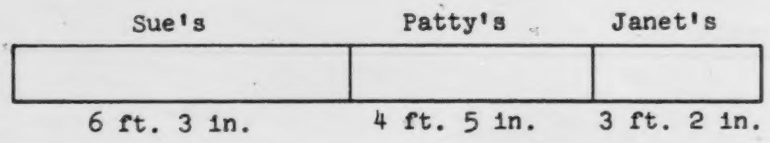
How would you use more than one unit to measure the door in your classroom? (Use feet and inches)

2. Now we will learn how to combine lengths.

Sue, Patty, and Janet are decorating a room for a party. They plan to put crepe paper around the windows. The girls had strips of paper with these lengths:

Sue	6 ft.	3 in.
Patty	4 ft.	5 in.
Janet	3 ft.	2 in.

a) If they put their strips of paper end to end it would look like this:



To find the length of the combined strips, we change all measures so that we use only one unit, the inch.

How many inches of paper did Sue have? (75 in.)

How many inches of paper did Patty have? (53 in.)

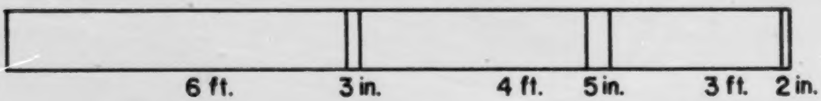
How many inches of paper did Janet have? (38 in.)

Now we can add the measures: 75 + 53 + 38 = 166.

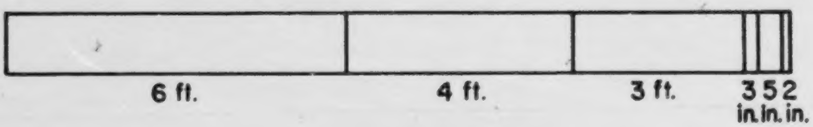
The length of the paper was 166 inches.

Express this length in feet and inches. (13 ft 10 in.)

b) We can show the combined strips by another sketch like this:



c) Now let us think about the combined strips in a different order. We can show our thinking by this sketch:



The answer could be found by adding the measures which go with the same unit.

6 + 4 + 3 = 13

6 ft.
4 ft.
3 ft.
13 ft.

3 + 5 + 2 = 10

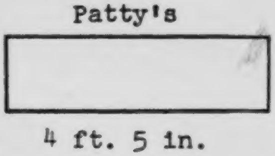
3 in.
5 in.
2 in.
10 in.

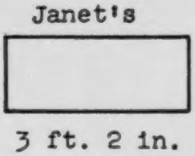
Usually we write:

6 ft. 3 in.
4 ft. 5 in.
3 ft. 2 in.
13 ft. 10 in.

The girls have 13 ft. 10 in. of crepe paper.

3. Which strip of paper is longer?





a) The length of Patty's paper, in inches 53 in.

The length of Janet's paper, in inches 38 in.

The lengths are in the same unit.

We subtract the measures: $53 - 38 = n$

$$\begin{array}{r} 53 \\ \underline{-\ 38} \\ 15 \end{array}$$

Patty's strip is 15 inches longer than Janet's.

How many feet and inches is this? (1 ft. 3 in.)

b) We can work the problem another way.

We subtract the measures which go with the same unit.

$$4 - 3 = 1 \qquad \begin{array}{ll} 4\ \text{ft.} & 5\ \text{in.} \\ \underline{3\ \text{ft.}} & \underline{2\ \text{in.}} \\ 1\ \text{ft.} & 3\ \text{in.} \end{array} \qquad 5 - 2 = 3$$

The work is usually written like this:

$$\begin{array}{ll} 4\ \text{ft.} & 5\ \text{in.} \\ \underline{3\ \text{ft.}} & \underline{2\ \text{in.}} \\ 1\ \text{ft.} & 3\ \text{in.} \end{array}$$

The length of Patty's paper is 1 ft. 3 in. more than the length of Janet's paper.

4. Martin, Charles, and John had a contest to see who could throw a football the farthest. See the record of their distances.

Martin	18 yd. 2 ft.
Charles	16 yd. 1 ft.
John	17 yd. 2 ft.

a) How much farther did Martin throw than Charles? (2 yd. 1 ft.)

b) How much farther did Martin throw than John? (1 yd. 0 ft.)

c) How many feet did Charles throw the ball? (49 ft.)

d) How far was the football thrown in three tries? (51 yd 5 ft. or 52 yd 2 ft)

Can we add the three measures?

We can add only the measures which have the same unit.

18	2
16	1
17	2
51	5

The football was thrown 51 yd. 5 ft by the boys.

5 ft. = 1 yd. 2 ft.

$$\begin{array}{r} 51 \text{ yd.} \phantom{\ 2 \text{ ft.}} \\ \underline{1 \text{ yd.}\ 2 \text{ ft.}} \\ 52 \text{ yd.}\ 2 \text{ ft.} \end{array}$$

51 yd. 5 ft. is the same measure as 52 yd. 2 ft.

Exercise Set 10

1. Dick has two pencils. One pencil is 7 inches long and one is 6 inches long. When put end to end his two pencils show a segment which measures (1) ft. (1) in.

2. Jane has a piece of ribbon 2 ft. 8 in. in length; Sara's ribbon is 3 ft. 9 in. long.

   a) How much shorter is Jane's ribbon than Sara's? (1 ft 1 in.)

   b) The two ribbons, when put end to end, make a piece 6 ft. 5 in. long.

3. Add these measures. (Write the lengths in the answers in two ways, as shown in exercise a).

a) 6 yd. 2 ft.
7 yd. 1 ft.
9 yd. 2 ft.
22 yd. 5 ft. = 23 yd. 2 ft.

b) 23 ft. 8 in.
35 ft. 2 in.
46 ft. 10 in.
(104 ft. 20 in. = 105 ft. 8 in.)

c) 38 yd. 2 ft.
23 yd. 2 ft.
66 yd. 2 ft.
(127 yd. 6 ft. = 129 yd. 0 ft.)

d) 11 yd. 2 ft.
37 yd. 1 ft.
9 yd. 2 ft.
(57 yd. 5 ft. = 58 yd. 2 ft.)

e) 24 ft. 2 in.
75 ft. 6 in.
67 ft. 6 in.
(166 ft. 14 in. = 167 ft. 2 in.)

f) 8 in.
5 in.
4 in.
(17 in. = 1 ft. 5 in.)

4. Subtract these measures.

a) 5 yd. 2 ft.
2 yd. 1 ft.
(3 yd. 1 ft.)

c) 33 yd. 2 ft.
8 yd. 2 ft.
(25 yd. 0 ft.)

b) 37 ft. 11 in.
18 ft. 8 in.
(15 ft. 3 in.)

d) 26 ft. 9 in.
9 ft. 4 in.
(17 ft. 5 in.)

5. Here are the names and heights (in inches) of 30 pupils of one classroom.

COLUMNS

	John 56	Mary 54	Jeannie 53	Royce 63	Tom 52	(278)
	Denise 53	Paul 61	Jack 63	Phyllis 52	Helen 52	(281)
ROWS	Fran 55	Fred 59	Martha 54	Walter 61	Viola 60	(289)
	Bill 52	Glen 52	Lenore 61	Gary 56	Boyd 65	(286)
	Sarah 52	Greg 52	Valaree 63	Jake 56	Gerry 65	(288)
	Edna 52	Will 54	James 52	Hill 65	Jo 64	(287)
	(320)	(332)	(346)	(353)	(358)	

a) Which row of pupils has the greatest total height? (3rd row)
How much is this greatest total height? (289 in. or 24 ft 1 in.)

b) Which column of pupils has the greatest total height? (5th column)
How much is this greatest total height? (358 in. or 29 ft 10 in.)

c) Is there one pupil who is sitting both in the row and in the column of greatest total height? (yes) Who? (Viola)

6. Use the table of exercise 5 to answer the following.

a. Which column of pupils has the least total height? (1st column)

How much is the least total height? (320 in or 26 ft 8 in.)

b. Which row of pupils has the least total height? (1st row)

How much is this total height? (278 in or 23 ft 2 in.)

c. Is there one pupil who is sitting both in the row and in the column of least total height? (yes)

Who? (John)

PERIMETER

Objective: To develop the following understandings and skills:

1. A simple closed curve may be measured by "straightening" a wire model of the curve and finding the length of the model.
2. A simple closed curve which is the union of line segments is a polygon.
3. The perimeter of a polygon is the length of the polygon.
4. The perimeter of a polygon may be computed by adding the measures of the sides, if the measures are expressed in terms of the same linear units.

Materials Needed:

Teacher: Wire, straightedge, scissors

Pupils: Piece of wire 15 inches long, ruler, another piece of wire about 20 inches long, scissors

Vocabulary: Perimeter

> In this section, pupils will measure a simple closed wire by bending a wire to the shape of a curve, straightening the wire, then measuring the wire to find the length of the curve. Perimeter is defined as the length of a polygon. The Explorations and Exercises which follow should lead to generalizations about computing perimeter.

P502

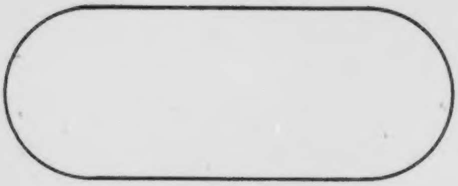
PERIMETER

Lengths of Simple Closed Curves

Exploration

1. Take a piece of wire, 15 inches long.

   Bend it to make a simple closed curve, like this:



   What is the length of the wire when it is bent in the shape of this curve? (15 in.)

2. Straighten out the wire, and bend it to form a different closed curve.

   What is the length of the new curve? (15 in.)

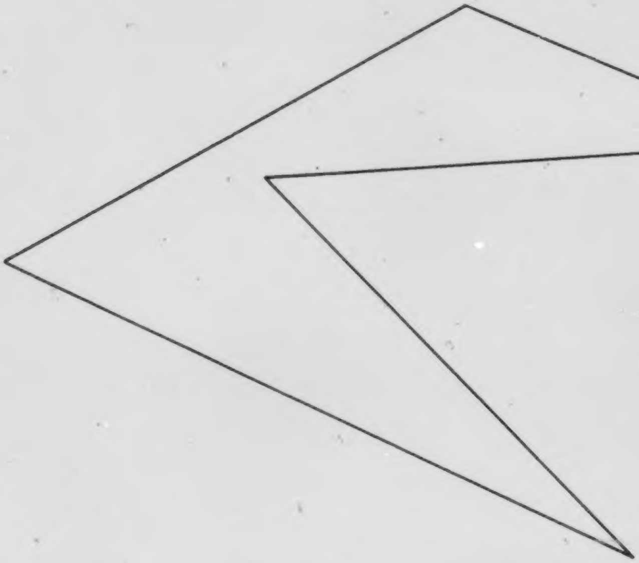
   What happens to the length of the wire when you change the shape of the simple closed curve? (The length of the wire remains the same.)

343

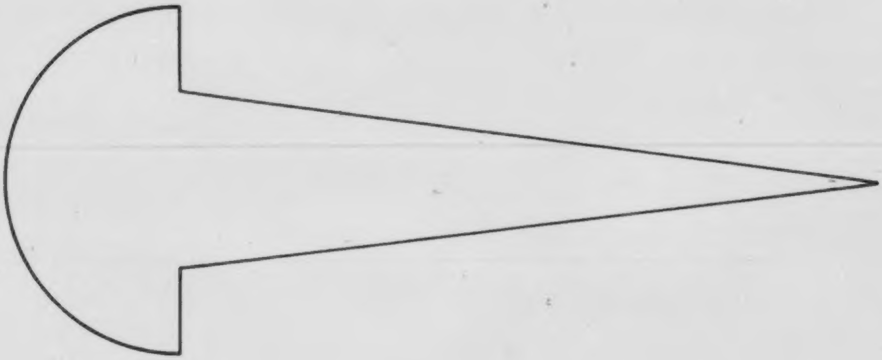
P503

3. Straighten out your wire.

Can you use it to find the length of the curve drawn below? (yes, 15 in.)



4. Is the length of the curve below, greater or less than 15 inches? (less than 15 in.)



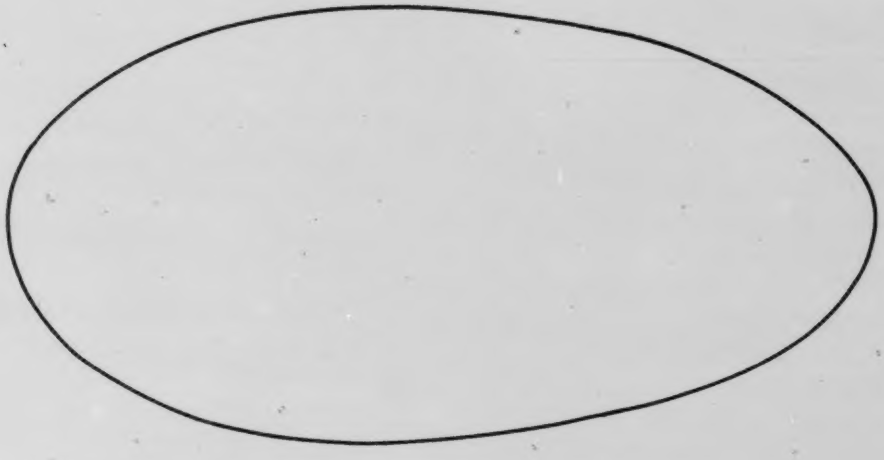
How much less than 15 inches? (about 3 inches)

Did you use your ruler to measure the length of the wire that remained after you bent the piece of wire to fit the curve above? (yes)

5. Take another piece of wire whose length you do not know.

Bend it to fit the curve below.

If there is wire left over, bend it back and out of the way.



Can you think of some way of using the wire outline to find the length of the curve? (Straighten out the wire outline and then measure the wire with a ruler.)

If you straighten the wire out, will the length of the part of the wire that outlined the curve change? (No)

Straighten out the wire.

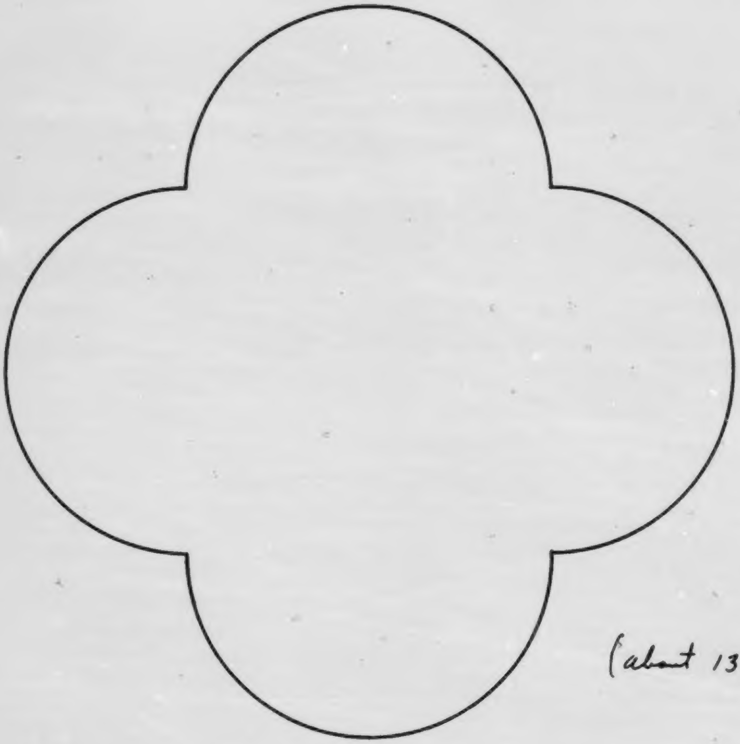
Measure it with a ruler.

What is the length of the curve? (about 12 inches)

Exercise Set 11

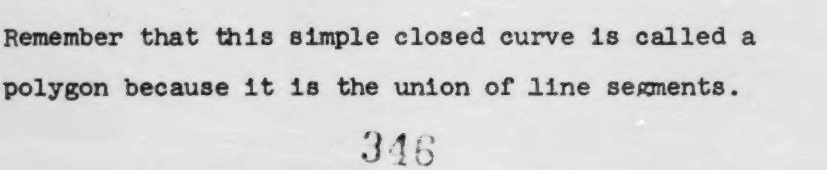
Use pieces of wire and a ruler to find the lengths of the curves.

1.



(about 13 inches)

2.



(about 10 inches)

Remember that this simple closed curve is called a polygon because it is the union of line segments.

3. Take a piece of wire. Bend it, using the whole piece of wire, so that you make an equilateral triangle.

   What is the length of the triangle? (It is the same length as the piece of wire from which it was made.)

4. Bend a piece of wire so that, using the whole piece of wire, you have a polygon with four sides of the same length.

   Find the length of the polygon. (The length of the polygon will be the same length as the piece of wire from which it was made.)

5. Use the same piece of wire as in exercise 4 and make a different polygon with four sides of the same length. Without straightening the wire, do you know the length of this polygon? (It is the same as it was in Exercise 4.)

6. Find a model of a circle in your home.

   Use a piece of wire or a piece of string to help you find the length of the circle.

   Did you need a ruler? (yes)

7. Cut a model of a triangle with its interior from a piece of cardboard.

   Can you find the length of the triangle? (yes)

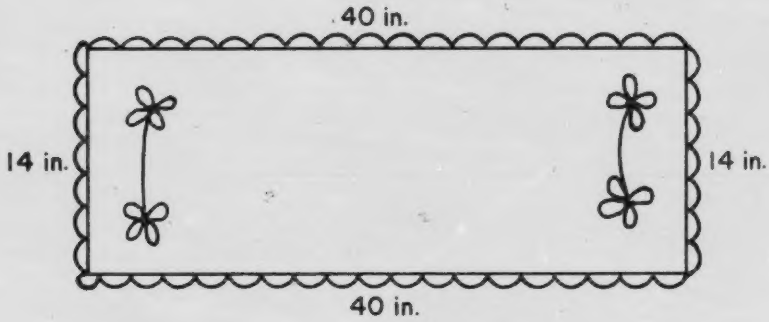
   Did you use a piece of wire? (yes or no)

   Could you have found the length of the triangle using the ruler only? (yes)

P507

PERIMETERS OF POLYGONS

Exploration



Joan wished to buy some lace edging to trim a scarf.

The scarf was 40 inches long and 14 inches wide.

How much edging does Joan need?

The number sentence which tells us the measure of the length of the edging Joan needs to trim all four edges of the scarf is:

$$40 + 14 + 40 + 14 = 108$$

The length of the edging that Joan must buy is 108 inches.

How many yards of edging does she need? (3 yards)

The perimeter of the rectangle in this example is 108 inches.

You have found the length of many simple closed curves.

When the curve is a polygon whose sides are line segments, we call the length of the polygon its perimeter.

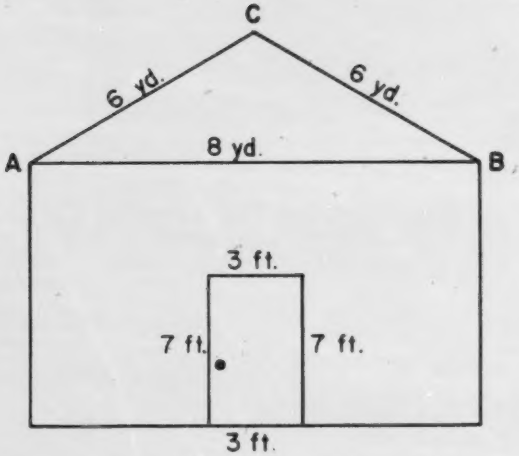
The perimeter gives a number and a unit of measure.

P508

2. The Jones family decided to decorate the front of their home for the Christmas season.

Johnny wanted to put a string of colored lights on the house along the triangle ABC. Mary wanted to put a string of colored lights around the doorway.

Mr. Jones said he would buy lights for the door or the roof. He would not buy lights for both. Also, he would decorate the one which required the shorter string of lights.



Johnny measured the 3 sides of the triangle.

Mary measured the 4 sides of the rectangle around the door.

Each reported that the sum of the measures of the sides of the figure he measured was 20.

Johnny said: "6 + 6 + 8 = 20"

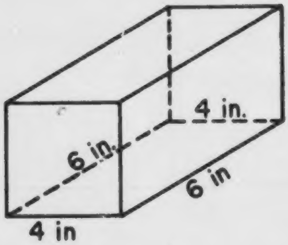
Mary said: "3 + 7 + 3 + 7 = 20"

What other fact did Mr. Jones need to know before he could make a decision about which part of the house to decorate? (the unit of measure)

Did the Jones family decorate the door or the roof? (the door)

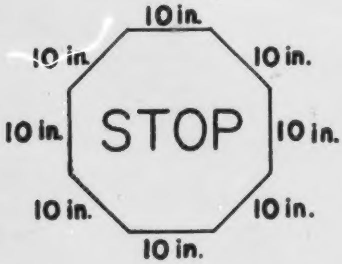
Exercise Set 12

1. Joe made a cardboard model of a chalkbox. He wished to tape the edges of the bottom of the box with scotch tape. How much scotch tape did he need? (20 in.) (There is no overlap at the corners.)



Do the edges of the bottom make a rectangle? (yes)
Did you find the sum of the measures of the sides of the rectangle, or did you find the perimeter of the rectangle? (the perimeter)

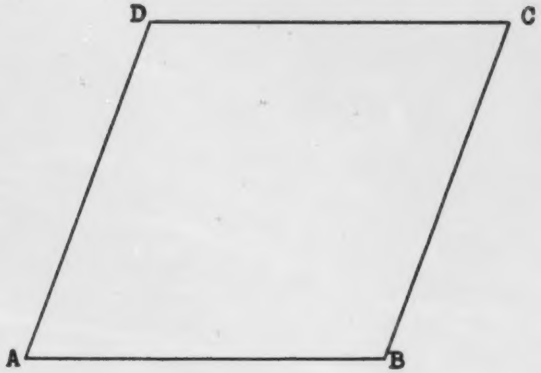
2. The police department of a town is painting a thin black border around the edge of the STOP signs. How many inches of border must be painted on each sign? (80 in.)



The edge of the STOP sign represents a polygon.
What is the perimeter of this polygon? (80 in.)

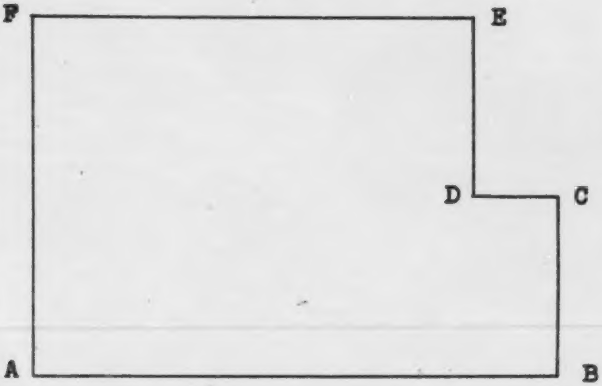
3. Use your ruler to find, to the nearest inch, the measure you need to find the perimeter of each polygon.

a)



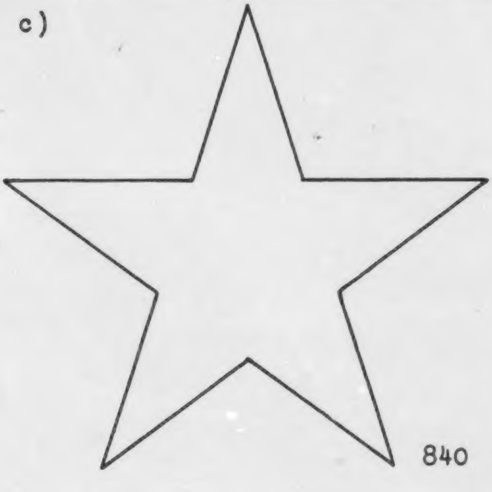
The perimeter
of figure ABCD
is (8 in).

b)



The perimeter
of figure ABCDEF
is (10 in).

c)



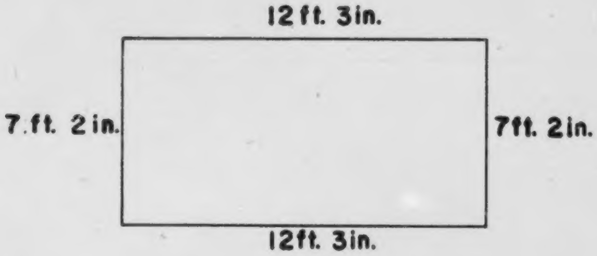
The perimeter
of the star
is (10 in).

FINDING PERIMETERS

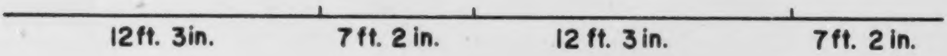
Exploration

Here is a plan for the floor of a rectangular room.

What is the perimeter of the edge of the floor?

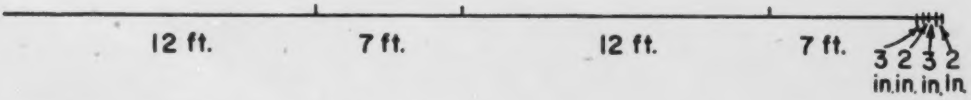

12 ft. 3in.
7 ft. 2 in.
7ft. 2in.
12ft. 3in.

If we place a piece of wire around the edge of this floor plan, then straighten out the wire, we can picture the wire like this:


12ft. 3in.
7 ft. 2 in.
12 ft. 3 in.
7 ft. 2 in.

Imagine that we cut the wire into eight pieces.

We put them together again, as shown in this picture.


12 ft.
7 ft.
12 ft.
7 ft.
3 in.
2 in.
3 in.
2 in.

Has the length changed? (n.)

We add the measures which have the same unit.

12 + 7 + 12 + 7 = 38

3 + 2 + 3 + 2 = 10

The length, or the perimeter, of the rectangle is 38 ft. 10 in.

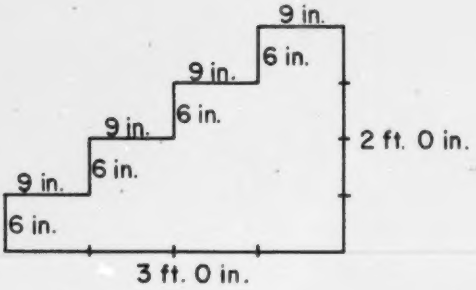
Remember we add the measures. We do not add the units. We add only those measures that were made using the same unit.

Exercise Set 13

1. Find the perimeter of a corner piece of land in the shape of a triangle if the lengths of its sides are 50 feet 4 inches, 80 feet 7 inches, and 50 feet 4 inches. (181 ft. 3 in. or 60 yd. 1 ft. 3 in.)

2. A yardstick is 1 inch wide. Find the perimeter of the face of the yardstick that shows the scale. Give your answer in yards and inches. (2 yd. 2 in.) Give your answer again in feet and inches. (6 ft. 2 in.)

3. Find the perimeter of the polygon pictured below:



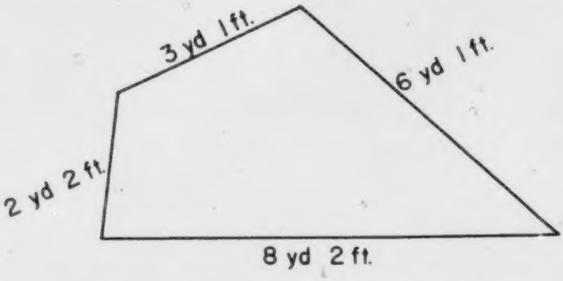
a) Can your answer be written using only one unit? (yes)
Write your answer in inches only. (120 in.) In feet only. (10 ft.)

b) Can your answer be written using yards only and what we have learned so far? (No.)

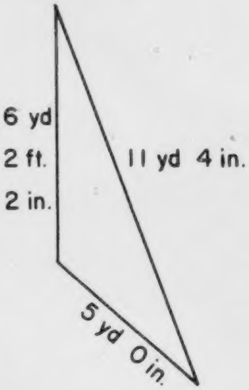
4. In France, a baseball diamond is a square, each of whose sides is 27 meters 70 cm. long. Pierre hits a home run. What is the length of the shortest path he can take, if he touches each base on his way back to home place? (110 meters 80 cm.) You will need to know that 100 centimeters equals 1 meter.

5. Which curve is longer? (Curve b) How much longer? 1 yd. 2 ft. 6 in.

a)



b)



BRAINTWISTERS

6. A man wants to put a fence all around his land. He knows that the boundary of his land can be thought of as a square. He measures one side and finds that it is between 125 and 126 feet long. How much fencing should he buy? (4 x 126 = 504 ft. at least)

7. In Jim's house a piece of glass in a window was broken. His father measured the frame where the glass went. He found that it was a rectangle with 14 inch and 24 inch sides, to the nearest inch. He bought a new piece of glass 14 inches wide and 24 inches long. When he went to put it in the frame he found that it was too long to fit. Can you give a possible reason? (The frame was shorter than 24 inches. Jim's father should have been careful ? to buy a piece of glass no larger than the space in the frame.)

Chapter 10

CONCEPT OF RATIONAL NUMBERS

PURPOSE OF UNIT

Children can be expected to have some concept of rational numbers. Undoubtedly, they and many teachers have called numbers, such as $\frac{1}{2}$, $\frac{3}{4}$, $\frac{1}{4}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{4}{8}$, and so on, fractions or fractional numbers. In this chapter and in the work that follows, we will call this set of numbers indicated by pairs of whole numbers (where the second of each pair is not zero) the rational numbers. Really, we will be concerned with only a subset of the rational numbers--a set which sometimes is referred to as the rational numbers of arithmetic. This kind of number is suggested by measurement.

More specifically, the purpose of this chapter is to continue with the development of the concept of rational numbers in such a way as:

1. to emphasize the common aspects of situations involving measurement, noting that partitioning unit segments and regions into congruent parts and sets of objects into equivalent subsets provide appropriate models for rational numbers.
2. to develop a way of naming rational numbers using symbols called fractions and mixed forms. (The later consists of a whole number numeral and a fraction. For example, $4\frac{2}{5}$ and $2\frac{1}{2}$ are mixed forms. We have, in the past, called these "mixed numbers". Another numeral, decimal, will be used in the next grade.)
3. to help children learn that different numerals may indicate the same rational number.
4. to help children learn that the set of whole numbers is a subset of the rational numbers.

A Note to Teachers

Terminology used in this chapter and in the chapters on rational numbers that follow in the SMSG Mathematics For the Elementary School will associate the following ideas with these words:

1. rational number - an infinite set of ordered pairs of whole numbers where the second member of the pair is not zero.

   Any member of a specific set of ordered pairs may be used to indicate that specific rational number.

2. fraction - a symbol which names a rational number. $\frac{1}{2}$, $\frac{4}{5}$, $\frac{9}{5}$, and $\frac{150}{10}$, are fractions. The set of fractions $\left\{\frac{1}{2}, \frac{2}{4}, \frac{4}{8}, \frac{3}{6}, \ldots\right\}$ names the same rational number. We can write $\frac{1}{2} = \frac{2}{4} = \frac{4}{8}$, etc.

   Each fraction is determined by an ordered pair of numbers. The first number of the pair is called the numerator and is the number of congruent parts with which one is concerned. The second number of the pair, called the denominator, is the number of congruent parts into which a unit (region, segment) is partitioned.

3. decimal - symbol which can name a rational number when the second member of the pair is 10 or a power of 10, that is 100 or 10^2, 1000, or 10^3, etc. The second member, or the denominator, can be denoted by place value.

4. mixed form - a symbol which is a "combination" of a numeral for a whole number and a fraction. (You no doubt know it as a "mixed number", but, of course, numbers cannot be mixed.

MATHEMATICAL BACKGROUND

Introduction

In the study of mathematics in the elementary school, a child learns to use several sets of numbers. The first of these is the set of counting numbers, 1, 2, 3, 4, ... The second is the set of whole numbers, 0, 1, 2, 3, 4, ... The child also may have learned certain properties of whole numbers.

During the primary and middle grades the idea of "number" is enlarged, so that by the end of the sixth grade the child recognizes each of the following as a name for a number:

$$\frac{4}{3},\ \frac{1}{2},\ 3.6,\ 2\frac{1}{2},\ 8,\ 0,\ \frac{5}{5},\ \frac{6}{2},\ .01$$

In traditional language, we might say that when the child has completed the first six years of school mathematics he knows about "the whole numbers, fractions, decimals, and mixed numbers." This language is primarily numeral language. It obscures the fact that a single number can have names of many kinds. "Fractions, decimals, and mixed numbers" are kinds of number names rather than different kinds of numbers. Whether we make a piece of ribbon $1\frac{1}{2}$ in. long, or 1.5 in. long, or $\frac{3}{2}$ in. long makes no difference--our ribbon is the same whatever our choice of numeral. That is, $1\frac{1}{2}$, 1.5, $\frac{3}{2}$ are all names for the same number. This number is a member of a set of numbers sometimes called the non-negative numbers or the rational numbers of arithmetic. For our purposes here, we shall call them the rational numbers, realizing that they are only a subset of the set of all rational numbers. It also should be realized that within the set of rational numbers is a set which corresponds to the set of whole numbers. For example, 0, 3, 7 are all rational numbers that are also whole numbers. $\frac{3}{4}$, $\frac{7}{4}$, and .2 are rational numbers that are not whole numbers.

First Ideas About Rational Numbers

Children develop early ideas about rational numbers by working with regions--rectangular regions, circular regions, triangular regions, etc. In Figures A, B, and C, rectangular regions have been used. For any type of region we must first identify the unit region. In Figures A, B, and C, the unit region is a square region.

In Figures A and B we see that:

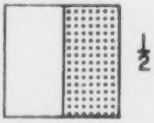
(1) The unit region has been separated into a number of congruent regions.

(2) Some of the regions have been shaded.

(a) Using regions. Let us see how children use regions to develop their first ideas of rational numbers. The child learns in simple cases to associate a number like $\frac{1}{2}$ or $\frac{2}{3}$ with a shaded portion of the figure. (Rational numbers can also be associated with the unshaded portions.)

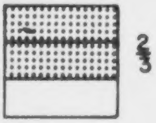
Using two or more congruent regions (Fig. C), he can separate each into the same number of congruent parts and shade some of the parts. Again, he can associate a number with the resulting shaded region.

Fig. A



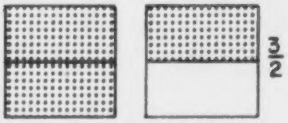
The unit square is separated into 2 congruent regions. 1 is shaded.

Fig. B



The unit square is separated into 3 congruent regions. 2 are shaded.

Fig. C



Each unit square is separated into 2 congruent regions. 3 are shaded. We have $\frac{3}{2}$ of a unit square.

At this point, the child is only at the beginning of his concept of rational numbers. However, let us note what we are doing when we introduce, for example $\frac{2}{3}$. We separate the (unit) region into 3 congruent parts. Then we shade 2 of these parts. Similarly, in $\frac{3}{2}$, we separate each (unit) region into 2 congruent regions, and shade 3 parts. In using regions to represent a number like $\frac{3}{2}$, we must emphasize the fact that we are thinking of $\frac{3}{2}$ of a unit region, as in Fig. C.

(b) Using the number line. The steps used with regions can be carried out on the number line. It is easy to see that this is a very practical thing to do. If we have a ruler marked only in inches, we cannot make certain types of useful measurements. We need to have points between the unit intervals, and we would like to have numbers associated with these points.

The way we locate new points on the ruler parallels the procedure we followed with regions. We mark off each unit segment into congruent parts. We count off these parts. Thus, in order to locate the point corresponding to $\frac{2}{3}$, we must mark off the unit segment in 3 congruent parts. We then count off 2 of them. (Fig. D) If we have separated each unit interval in 2 congruent parts and counted off 3 of them, we have located the point which we would associate with $\frac{3}{2}$. (Fig. E)

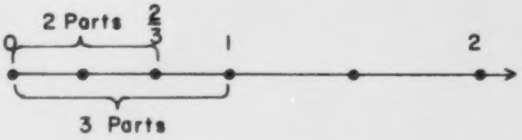


Fig. D

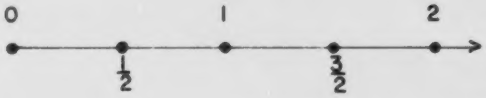


Fig. E

Once we have this construction in mind, we see that all such numbers as $\frac{3}{4}$, $\frac{5}{8}$, $\frac{2}{4}$, $\frac{1}{3}$, $\frac{4}{3}$, $\frac{11}{8}$ can be associated with particular points on the number line. To locate $\frac{11}{8}$, for example, we mark the unit segments into 8 congruent segments. (Fig. F)

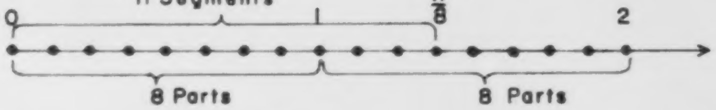


Fig. F

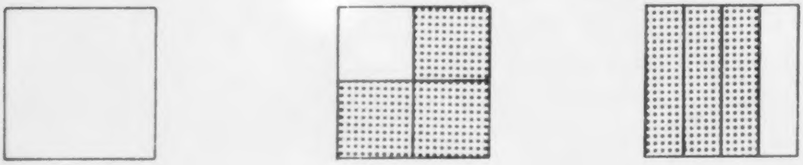
(c) Numerals for pairs of numbers. Suppose that we consider a pair of counting numbers such as 11 and 8 where 11 is the first number and 8 is the second number. We can make a symbol, writing the name of the first number of the pair above the line and that of the second below. Thus for the pair of numbers, 11 and 8, our symbol would be $\frac{11}{8}$. If we had thought of 8 as the first number of the pair and 11 as the second, we would have said the pair 8 and 11, and the symbol would have been $\frac{8}{11}$. For the numbers 3 and 4, the symbol would be $\frac{3}{4}$. For the numbers 4 and 3, the symbol would be $\frac{4}{3}$.

With the symbol described in the preceding paragraph, we can associate a point on the number line. The second number tells into how many congruent segments to separate each unit segment. The first number tells how many segments to count off.

We also can associate each of our symbols with a shaded region as in Fig. A, B, and C. The second number tells us into how many congruent parts we must separate each unit region. The first number tells us how many of these parts to shade.

For young children, regions are easier to see and to work with than segments. However, the number line has one strong advantage. For example, we associate a number as $\frac{3}{4}$, with exactly one point on the number line. The number line also gives an unambiguous picture for numbers like $\frac{3}{2}$ and $\frac{7}{2}$. A region corresponding to $\frac{3}{4}$ is less precisely defined in that regions with the same measure need not be identical or even congruent.

In Fig. G, we can see that each shaded region is $\frac{3}{4}$ of a unit square. Recognizing that both shaded regions have $\frac{3}{4}$ sq. units is indeed one part of the area concept.



Unit square

Figure G

When we match numbers with points on the number line, we work with segments that begin at 0. For this reason, though the number line is less intuitive at early stages, it is well to use it as soon as possible.

Meaning of Rational Number

The diagrams Fig. H, (a), (b), (c), show a number line on which we have located points corresponding to $\frac{1}{2}$, $\frac{2}{2}$, $\frac{3}{2}$, etc. and a number line on which we have located points corresponding to $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$, etc. Also shown, is a number line with $\frac{1}{8}$, $\frac{2}{8}$, etc. As we look at these lines, we see that it seems very natural to think of $\frac{0}{2}$ as being associated with the 0 point. We are really, so to speak, counting off 0 segments. Similarly, it seems natural to locate $\frac{0}{4}$ and $\frac{0}{8}$ as indicated.

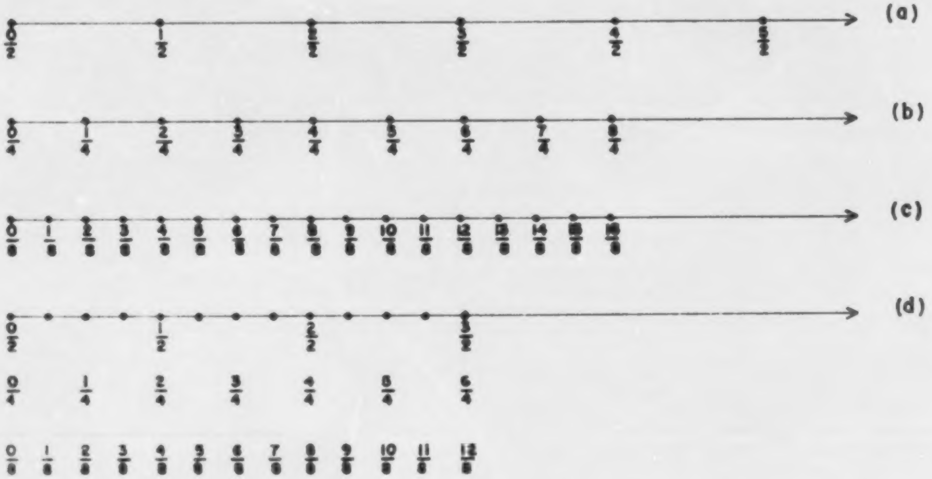

(a)
(b)
(c)
(d)
0/2 1/2 2/2 3/2 4/2 5/2
0/4 1/4 2/4 3/4 4/4 5/4 6/4 7/4 8/4
0/8 1/8 2/8 3/8 4/8 5/8 6/8 7/8 8/8 9/8 10/8 11/8 12/8 13/8 14/8 15/8 16/8
0/2 1/2 2/2 3/2
0/4 1/4 2/4 3/4 4/4 5/4 6/4
0/8 1/8 2/8 3/8 4/8 5/8 6/8 7/8 8/8 9/8 10/8 11/8 12/8

Figure H

Now let us put our diagrams (a), (b), (c) together. In other words, let us carry out on a single line (d) the process for locating all the points.

When we do this, we see that $\frac{1}{2}$, $\frac{2}{4}$, and $\frac{4}{8}$ are all associated with the same point. In the same way, $\frac{3}{4}$ and $\frac{6}{8}$ are associated with the same point.

Now we are ready to explain more precisely what we mean by fraction and by rational number. Let us agree to call the symbols we have been using fractions. A fraction, then, is a symbol associated with a pair of numbers. The first number of the pair is called the numerator and the second number is called the denominator. So far, we have used only those fractions in which the numerator of the number pair is a whole number (0, 1, 2, ...), and the denominator is a counting number (1, 2, 3, ...).

Each fraction can be used to locate a point on the number line. To each point located by a fraction there corresponds a rational number. Thus, a fraction names the rational number. For example, if we are told the fraction $\frac{3}{10}$, we can locate a point that corresponds to it on the number line. $\frac{3}{10}$ is the name of the rational number associated with this point. This point, however, can also be located by means of other fractions, such as $\frac{6}{20}$ and $\frac{9}{30}$. Thus, $\frac{6}{20}$ and $\frac{9}{30}$ also are names for the rational number named by $\frac{3}{10}$ since they are associated with the same point. Rational numbers, then, are named by fractions of the type we have been discussing. To each point on the number line that can be located by a fraction, there corresponds a non-negative rational number.

A very unusual child might wonder whether every point on the number line can be located by a fraction of the kind we have described. We must answer "No". There are numbers--π being one of them and $\sqrt{2}$ being another--that have no fraction names of the sort we have described. Introducing such irrational numbers is deferred until the seventh and eighth grades.

The Whole Numbers As Rational Numbers

Our pattern for matching fractions with points on the number line can be used with these fractions: $\frac{0}{1}$, $\frac{1}{1}$, $\frac{2}{1}$, $\frac{3}{1}$, etc.

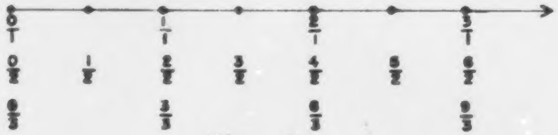


Fig. I

On the number line we see (Fig. I) that we matched $\frac{1}{1}$, $\frac{2}{2}$, $\frac{3}{3}$ with the same point. We note that this point is also matched with the counting number 1. Thus, to the same point corresponds

(1) the counting number 1

(2) the rational number named by $\frac{1}{1}$.

It seems that it would be a convenience to use the symbol 1 as still another name for the rational number named by $\frac{1}{1}$, $\frac{2}{2}$, etc. This would allow us to write $1 = \frac{2}{2}$, for example. In the same way, we would think of 5 as another name for the number named by $\frac{5}{1}$, $\frac{10}{2}$, etc.

We need at this point to be a little careful in our thinking. There is nothing illogical about using any symbol we like as a numeral. A problem *does* arise, however, when a single symbol has *two* meanings, because then we are in obvious danger that inconsistencies may result. For example, when we think of 2, 3, and 6 as *counting* numbers we are accustomed to writing $2 \times 3 = 6$. We will eventually define the product of two rational numbers, and we would be in serious trouble if the product of the *rational* numbers named by 2 and 3 were anything but the *rational* number named by 6.

However, using 0, 1, 2, 3, etc., as names for rational numbers never leads us into any inconsistency. For all the purposes of arithmetic--that is, for finding sums, products, etc., and for comparing sizes, we get names for whole numbers or names for rational numbers. In more sophisticated mathematical terms, we can say that the set of rational numbers contains a subset--those named by $\frac{0}{1}$, $\frac{1}{1}$, $\frac{2}{1}$, etc.--isomorphic to the set of whole numbers, that is $\frac{0}{1}$, $\frac{1}{1}$, etc. behave just like whole numbers, 0, 1, etc.

It would be overambitious to attempt to formulate the idea of isomorphism precisely in our teaching. It is sufficient for our purposes to regard 0, 1, 2, etc., as names for rational numbers. It is appropriate to note, however, in connection with operations on rationals, that where the operations are applied to numbers like $\frac{1}{1}$, $\frac{2}{1}$ they lead to results already known from experience with whole numbers.

Identifying Fractions That Name The Same Rational Number

When we write $\frac{1}{2} = \frac{3}{6}$, we are saying "$\frac{1}{2}$ and $\frac{3}{6}$ are names for the same number."

(a) Using physical models. The truth of the sentence $\frac{1}{2} = \frac{3}{6}$ can be discovered by concrete experience. In Fig. J, for example, we have first separated our unit region into two congruent regions. We have then separated each of these parts further into 3 congruent regions as shown in the second drawing. The second unit square is thus separated into 2×3, or 6 parts. Shading 1 part in the first drawing is equivalent to shading 1×3, or 3 parts in the second. We thus recognize that $\frac{1}{2} = \frac{1 \times 3}{2 \times 3}$.

Shading $\frac{1}{2}$ and $\frac{3}{6}$ of a region.

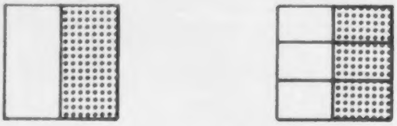


Fig. J

Again, our analysis of regions follows a pattern that can be applied on the number line. Let us consider $\frac{1}{2}$ and $\frac{4}{8}$.

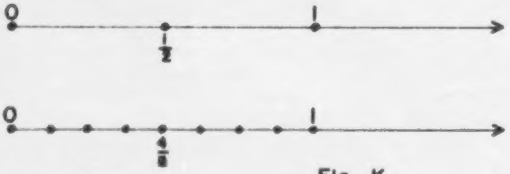


Fig. K

In locating $\frac{1}{2}$ on the number line, (Fig. K) we separate the unit interval into 2 congruent segments. In locating $\frac{4}{8}$, we separate it into 8 congruent segments. We can do this by first separating into 2 parts and then separating each of these 2 segments into 4 segments. This process yields (2×4) congruent segments. Taking 1 of 2 congruent parts thus leads to the same point as taking 4 of 8 congruent parts:

$$\frac{1}{2} = \frac{1 \times 4}{2 \times 4}$$

In other words, when we multiply the numerator and denominator of $\frac{1}{2}$ by the same counting number, we can visualize the result using the number line. We have subdivided our $\frac{1}{2}$ intervals into a number of congruent parts.

After many such experiences, children should be able to make a picture to explain this type of relationship. For example, region and number line pictures for $\frac{3}{4} = \frac{3 \times 2}{4 \times 2}$ are shown in Fig. L.

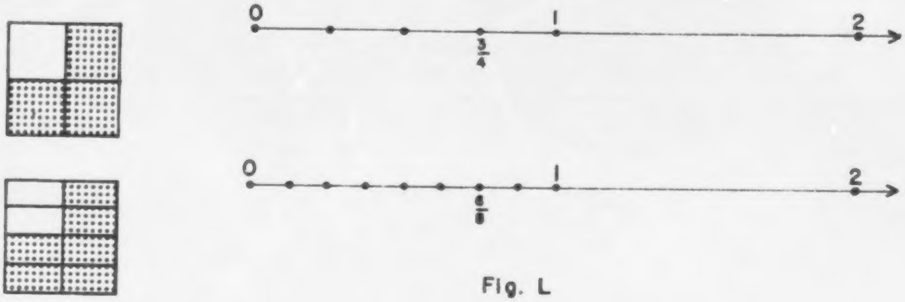


Fig. L

Each $\frac{1}{4}$ part (region or interval) is subdivided into 2 congruent parts; hence $\frac{3}{4} = \frac{3 \times 2}{4 \times 2}$.

(b) Using numerators and denominators. In a discussion about two fractions naming the same number, it may appear startling to emphasize multiplying numerator and denominator by the same counting number. We usually think about finding the simplest fraction name if we can. We think, then, $\frac{4}{8} = \frac{1}{2}$. But, of course "=" means "names the same number." Seeing $\frac{1}{2} = \frac{4}{8}$, we can think, $\frac{4}{8} = \frac{1}{2}$, and this will be particularly easy if the "names the same number" idea has been emphasized adequately.

Another familiar idea also is contained in what has been said. We often think about dividing numerator and denominator by the same counting number. For example, we think:

$$\frac{6}{8} = \frac{6 \div 2}{8 \div 2} = \frac{3}{4}$$

This is easy to translate into a multiplicative statement, since multiplication and division are inverse operations: $6 \div 2 = 3$ means $3 \times 2 = 6$.

(c) Using factoring. The idea that multiplying the numerator and denominator of a fraction by a counting number gives a new fraction that names the same number as the original fraction is an idea very well suited to the discussion in the unit on factoring. To find a simpler name for $\frac{12}{15}$, we write:

$$\frac{12}{15} = \frac{2 \times 2 \times 3}{5 \times 3} = \frac{2 \times 2}{5} = \frac{4}{5}$$

Suppose we are thinking about two fractions. How will we decide whether or not they name the same number? There are two possibilities.

Rule (1). It may be that for such fractions as $\frac{1}{2}$ and $\frac{2}{4}$, one fraction is obtained by multiplying the numerator and denominator of the other by a counting number. In other words, it may be that we can picture the fractions as was just done. Since $\frac{2}{4} = \frac{2 \times 1}{2 \times 2}$, $\frac{2}{4}$ and $\frac{1}{2}$ belong to the same set--thus name the same number.

Rule (2). It may be that, we cannot use Rule 1 directly. For example, $\frac{2}{4}$ and $\frac{3}{6}$ cannot be compared directly by Rule 1. However, we can use Rule 1 to see that $\frac{2}{4} = \frac{1}{2}$ and $\frac{3}{6} = \frac{1}{2}$, and in this way, we see that $\frac{2}{4}$ and $\frac{3}{6}$ name the same number.

Notice that in comparing $\frac{2}{4}$ and $\frac{3}{6}$, we might have used Rule 1 and 2 in a different way. We might have recognized that:

$$\frac{2}{4} = \frac{2 \times 3}{4 \times 3} = \frac{6}{12} \qquad \text{and} \qquad \frac{3}{6} = \frac{3 \times 2}{6 \times 2} = \frac{6}{12}.$$

or we might have said:

$$\frac{2}{4} = \frac{2 \times 6}{4 \times 6} = \frac{12}{24} \qquad \text{and} \qquad \frac{3}{6} = \frac{3 \times 4}{6 \times 4} = \frac{12}{24}.$$

In the latter example, we have renamed $\frac{2}{4}$ and $\frac{3}{6}$, using fractions with denominator 4×6. Of course, we recognize that $4 \times 6 = 6 \times 4$. (Commutative Property)

In our example, we see that 24 is a common denominator for $\frac{2}{4}$ and $\frac{3}{6}$, though it is not the least common denominator. Nevertheless, one common denominator for two fractions is always the product of the two denominators.

(d) A special test. Let us now consider a special test for two fractions that name the same rational number. In our last example we used 6×4 as the common denominator for $\frac{2}{4}$ and $\frac{3}{6}$. Thus we had

$$\frac{2}{4} = \frac{2 \times 6}{4 \times 6} \qquad \text{and} \qquad \frac{3}{6} = \frac{3 \times 4}{6 \times 4}.$$

We could say: It is true that $\frac{2}{4} = \frac{3}{6}$, because the two resulting numerators--2×6 and 3×4--are equal, and the denominators are equal.

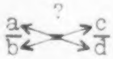
In other words, to test whether $\frac{2}{4} = \frac{3}{6}$, it is only necessary--once you have understood the reasoning--to test whether $2 \times 6 = 3 \times 4$. And this last number sentence is true!

In the same way, we can test whether $\frac{9}{15} = \frac{24}{40}$ by testing whether $9 \times 40 = 24 \times 15$. They do! When we do this, we are thinking:

$$\frac{9}{15} = \frac{9 \times 40}{15 \times 40} \qquad \text{and} \qquad \frac{24}{40} = \frac{24 \times 15}{40 \times 15}$$

This is an example of what is sometimes called "cross product rule." It is very useful in solving proportions. (Sometimes it is stated: The product of the means equals the product of the extremes.)

The rule states: To test whether two fractions $\frac{a}{b}$ and $\frac{c}{d}$ name the same number, we need only test whether $a \times d = b \times c$. That is,



This rule is important for later applications in mathematics such as similar triangles. In advanced texts on algebra, it is sometimes used as a way of defining rational numbers. That is, an advanced text might say: "A rational number is a set of symbols like $\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \ldots\}$. Two symbols, $\frac{a}{b}$ and $\frac{c}{d}$, belong to the same set if $a \times d = b \times c$."

What we have done amounts to the same thing, but is developed more intuitively. For teaching purposes, the "multiply numerator and denominator by the same counting number" idea conveyed by Rule 1 can be visualized more easily than can the "cross product" rule.

It would certainly not be our intention to insist that children learn Rules 1 and 2 formally. However, these rules summarize an experience that is appropriate for children. We can form a chain of fractions that name the same number,

$$\frac{1}{2} = \frac{2}{4} = \frac{4}{8} = \frac{8}{16} = \ldots .$$

Each fraction is formed by multiplying the numerator and denominator of the preceding one by 2. We can visualize this as subdividing repeatedly a segment or a region. (Rule 1). We can form a second chain beginning with $\frac{1}{2} = \frac{3}{6} = \frac{9}{18} \ldots$ We can then understand that it is possible to pick out any numeral from one chain and equate it with any numeral from the other, which is just what Rule 2 says.

Meaning of Rational Number - Summary

Let us summarize how far we have progressed in our development of the rational numbers.

(1) We regard a symbol like one of the following as naming a rational number:

$$\frac{3}{8}, \ \frac{0}{5}, \ \frac{7}{6}, \ 6, \ \frac{4}{3}, \ \frac{6}{4}, \ 1, \ \frac{5}{5}$$

(2) We know how to associate each such symbol with a point on the number line.

(3) We know that the same rational number may have many names that are fractions. Thus, $\frac{6}{4}$ and $\frac{3}{2}$ are fraction names for the same number.

(4) We know that when we have a rational number named by a fraction, we can multiply the numerator and denominator of the fraction by the same counting number to obtain a new fraction name for the same rational number.

(5) We know that in comparing two rational numbers it is useful to use fraction names that have the same denominators. We know, too, that for any two rational numbers, we can always find fraction names of this sort.

Thus far we have not stressed what is often called, in traditional language, "reducing fractions." To "reduce" $\frac{6}{8}$, for example, is simply to name it with the name using the smallest possible numbers for the numerator and the denominator. Since 2 is a factor both of 6 and 8, we see that

$$\frac{6}{8} = \frac{3 \times 2}{4 \times 2} = \frac{3}{4}$$

We have applied our general idea that "multiplying numerator and denominator by the same counting number" gives a new name for the same number. We can call $\frac{3}{4}$ the simplest name for the rational number it names.

We would say that we have found, in $\frac{3}{4}$, the simplest name for the rational number named by $\frac{6}{8}$. This is more precise than saying we have "reduced" $\frac{6}{8}$, since we have not made the rational number named by $\frac{6}{8}$ any smaller. We have used another pair of numbers to rename it.

(6) We know, also, that 2 and $\frac{2}{1}$ name the same number. We thus regard the set of whole numbers as a subset of the set of rational numbers. Any number in this subset has a fraction name with denominator 1. ($\frac{0}{1}$, $\frac{1}{1}$, $\frac{2}{1}$, etc. belong to this subset.) 2 is a name for a rational number which is a whole number. 2 is not a fraction name for this number, but the number has fraction names $\frac{2}{1}$, $\frac{4}{2}$, etc.

At this point, it seems reasonable to use "number" for rational numbers where the meaning is clear. We may ask for the number of inches or measure of a stick, or the number of hours in a school day.

(7) We can agree to speak of the number $\frac{2}{3}$, to avoid the wordiness of "number named by $\frac{2}{3}$." Thus, we might say that the number $\frac{2}{3}$ is greater than the number $\frac{1}{2}$ (as we can verify easily on the number line). This would be preferable to saying that "the fraction $\frac{2}{3}$ is greater than the fraction $\frac{1}{2}$," because we do not mean that one name is greater than another.

(8) We should not say that 3 is the denominator of the number $\frac{2}{3}$, because the same number has other names (like $\frac{4}{6}$) with different denominators. 3 is rather the denominator of the fraction $\frac{2}{3}$.

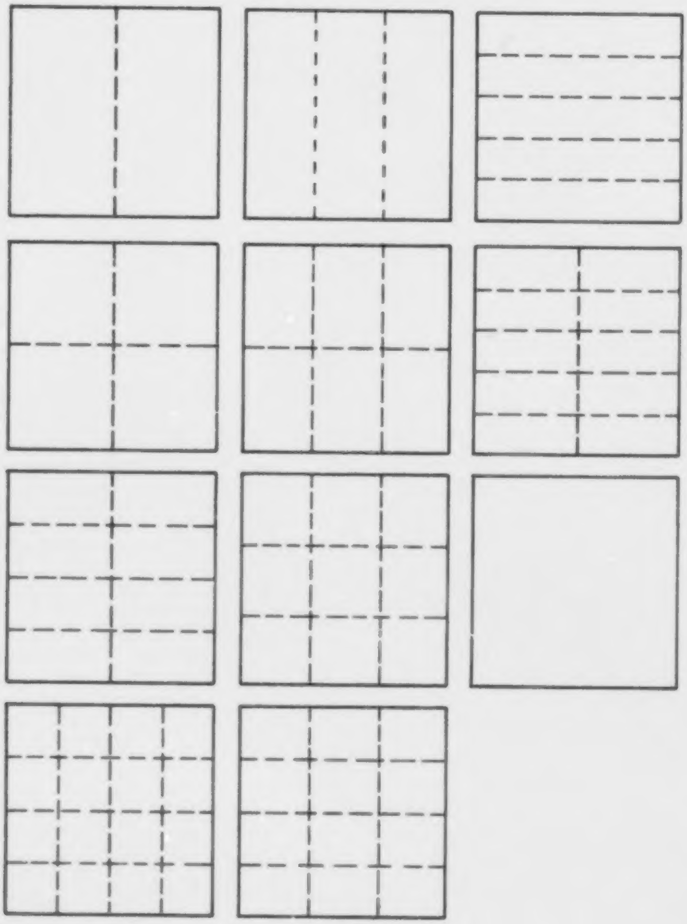
(9) We have seen that the idea of rational number is relevant both to regions and line segments. We will see soon how it relates to certain problems involving sets.

Now we might introduce some decimals. The numeral, .1, for example, is another name for $\frac{1}{10}$. However, we can explain a numeral like 1.7 more easily when we have developed the idea of adding rational numbers.

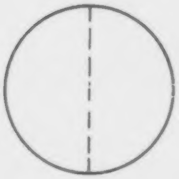
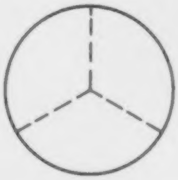
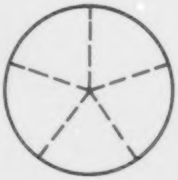
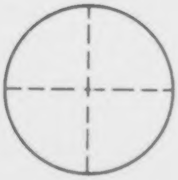
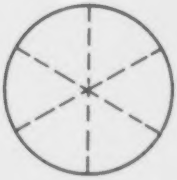
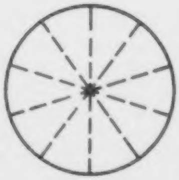
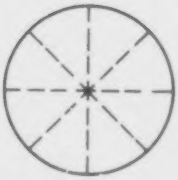
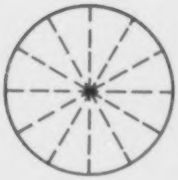
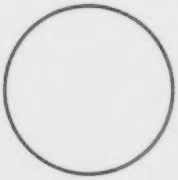
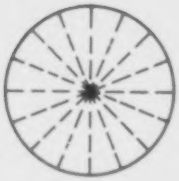
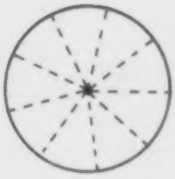
MATERIALS

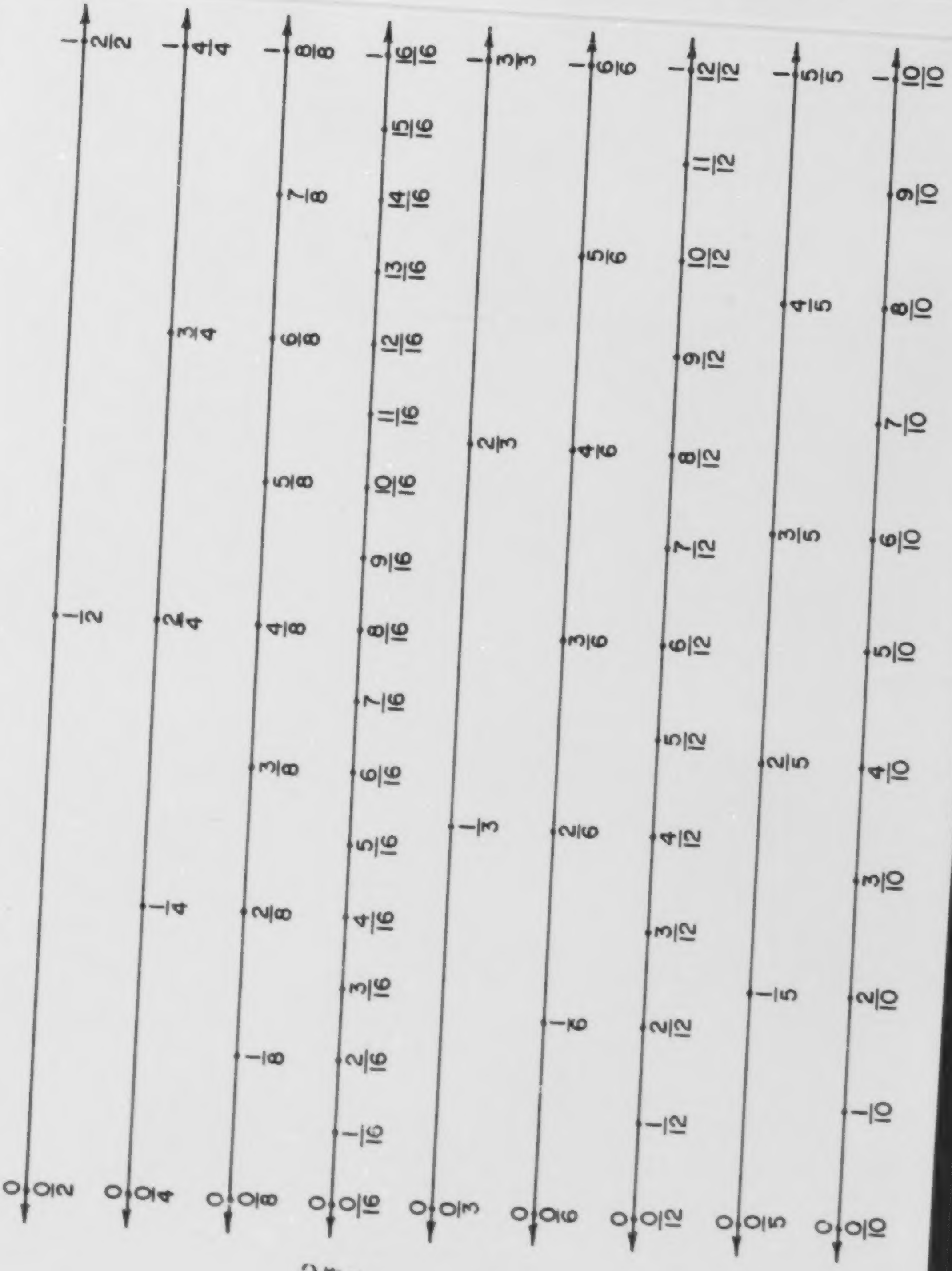
It is important that extensive use be made of materials in developing understanding of the rational numbers. Some materials which have been found useful are suggested on the next few pages. These may be supplemented by other available materials.

Teachers will find copies of these cards made on foot square cardboard for teacher models and smaller cards for each child useful throughout the chapter. Colored acetate may be used to indicate shaded areas on teacher models. Colored paper parts might be used by children to designate a specific number of parts of a region.



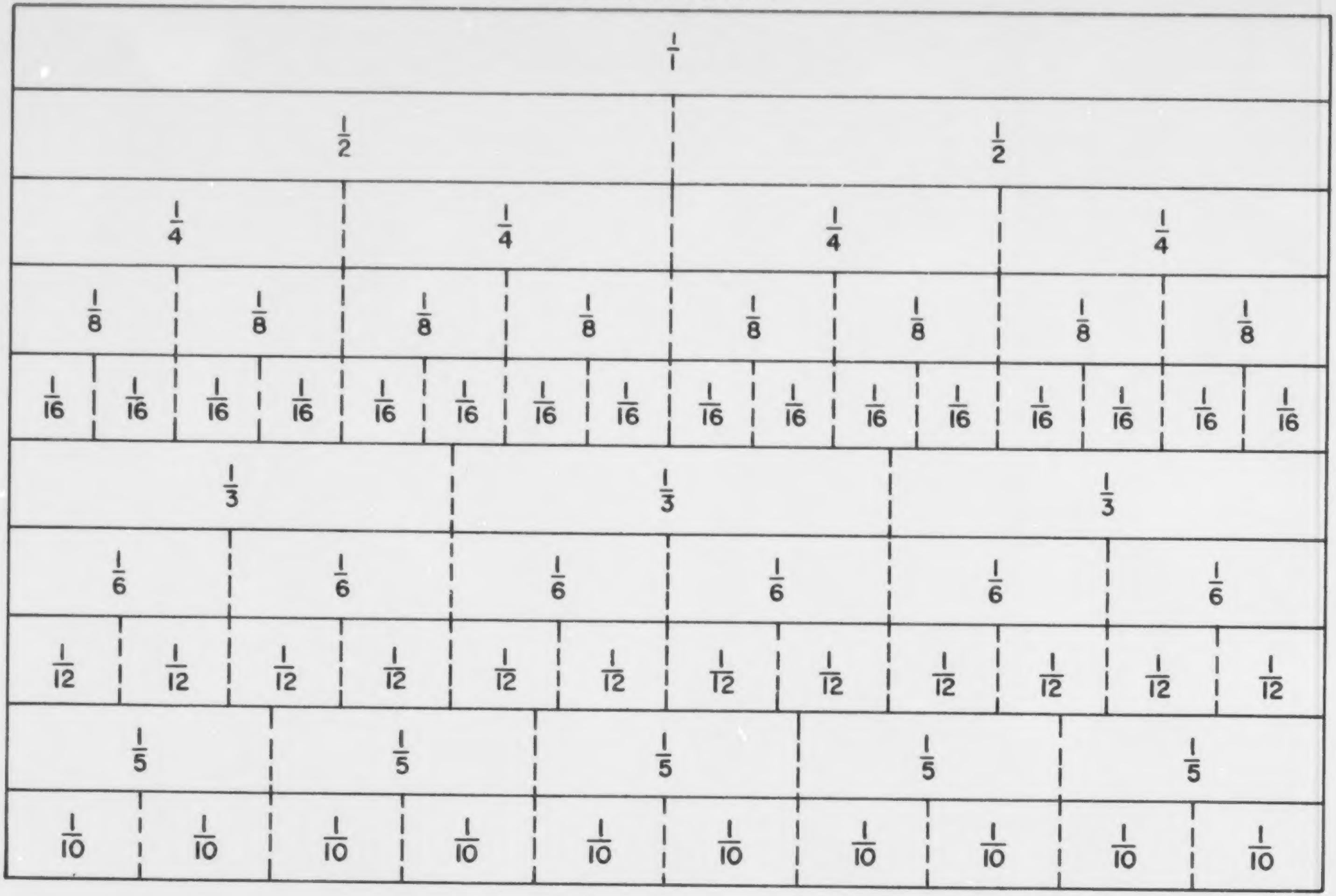
Models of circular regions can be copied on cardboard, construction paper, or undecorated paper plates.

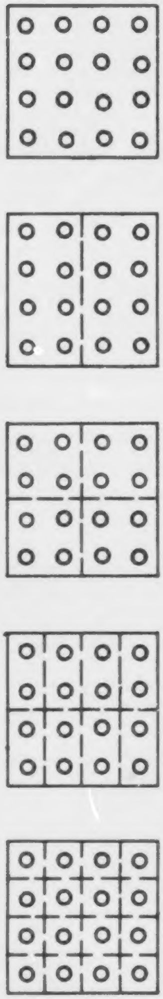














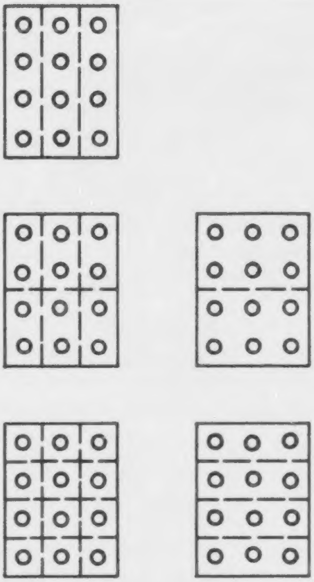
0 0/2 1/2 1 2/2
0 0/4 1/4 2/4 3/4 1 4/4
0 0/8 1/8 2/8 3/8 4/8 5/8 6/8 7/8 1 8/8
0 0/16 1/16 2/16 3/16 4/16 5/16 6/16 7/16 8/16 9/16 10/16 11/16 12/16 13/16 14/16 15/16 1 16/16
0 0/3 1/3 2/3 1 3/3
0 0/6 1/6 2/6 3/6 4/6 5/6 1 6/6
0 0/12 1/12 2/12 3/12 4/12 5/12 6/12 7/12 8/12 9/12 10/12 11/12 1 12/12
0 0/5 1/5 2/5 3/5 4/5 1 5/5
0 0/10 1/10 2/10 3/10 4/10 5/10 6/10 7/10 8/10 9/10 1 10/10

Fraction Chart

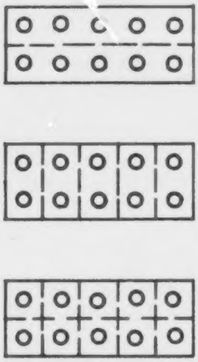

1/1
1/2 1/2
1/4 1/4 1/4 1/4
1/8 1/8 1/8 1/8 1/8 1/8 1/8 1/8
1/16 1/16 1/16 1/16 1/16 1/16 1/16 1/16 1/16 1/16 1/16 1/16 1/16 1/16 1/16 1/16
1/3 1/3 1/3
1/6 1/6 1/6 1/6 1/6 1/6
1/12 1/12 1/12 1/12 1/12 1/12 1/12 1/12 1/12 1/12 1/12 1/12
1/5 1/5 1/5 1/5 1/5
1/10 1/10 1/10 1/10 1/10 1/10 1/10 1/10 1/10 1/10



Set A



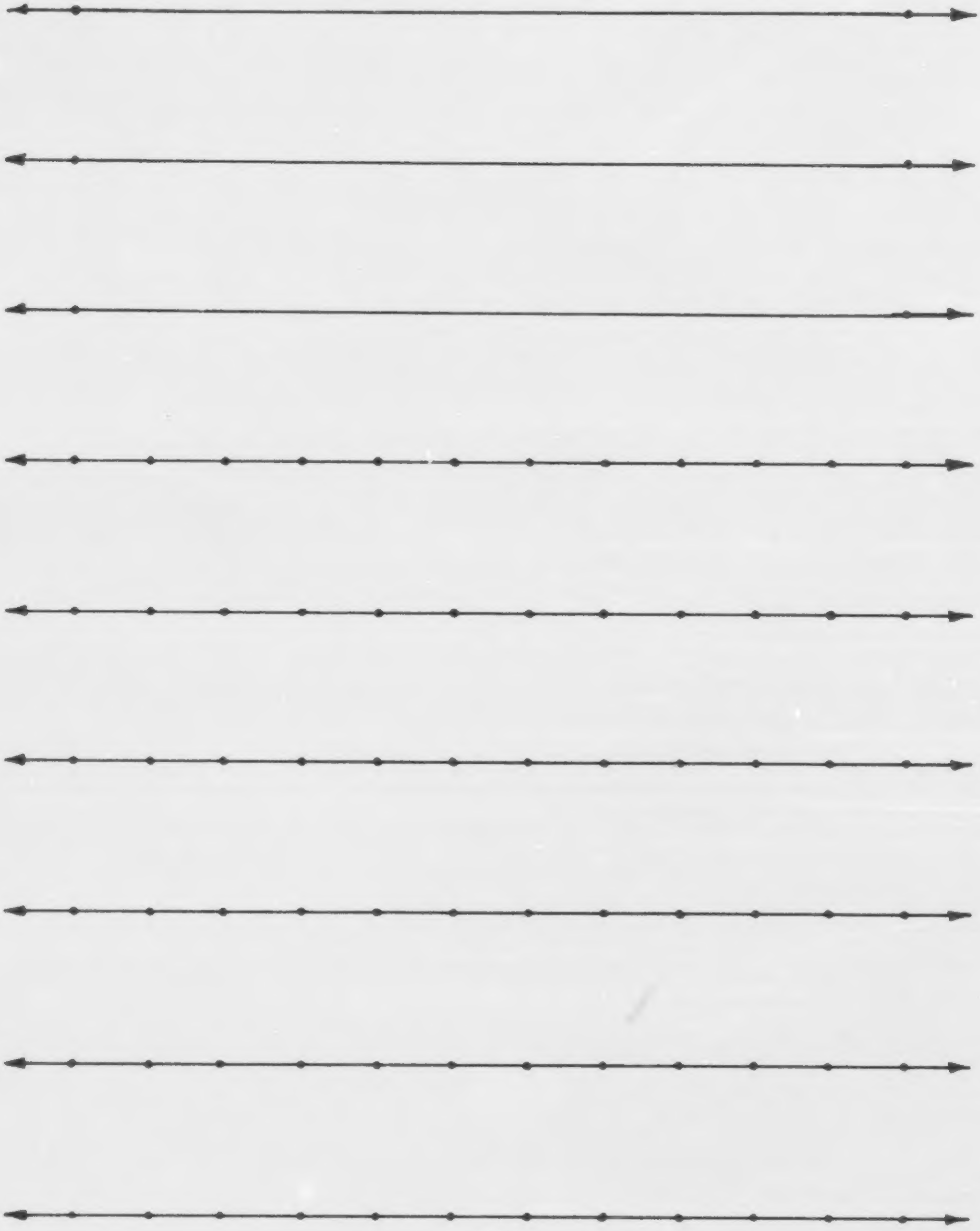
Set B



Set C

These arrays may be used to develop the concept of $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, etc., of a set of objects. Colored acetate may be used with them to indicate $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{6}$, etc., of a set of objects.

A sheet of number lines similar to this can be dittoed for children to use throughout unit.



IDEA OF RATIONAL NUMBERS

Objective: To provide experiences that suggest rational numbers without using any fraction symbols for the numbers, by

a) partitioning regions and line segments into congruent parts and sets of objects into equivalent subsets; and

b) naming the number of congruent parts into which a unit region or segment has been partitioned and the number of parts to be used.

Materials: See suggestions for materials to be used in this unit.

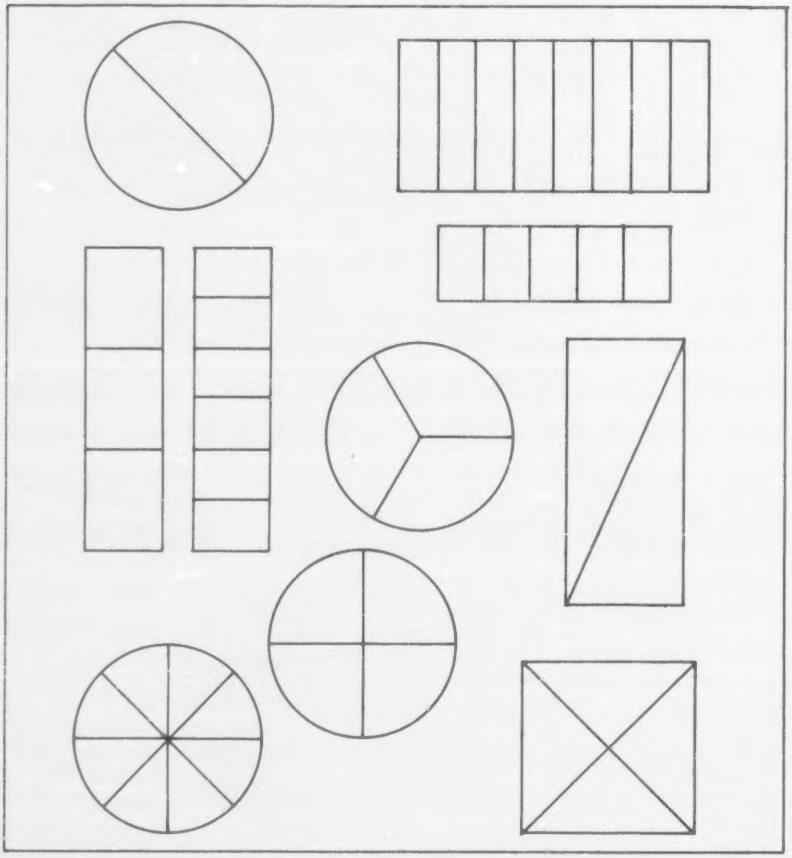
In these first experiences, which may be review, work will be limited to those situations in which the number of parts identified does not exceed the number of congruent parts into which the unit has been partitioned; i.e., $n \geqq 1$.

Teaching Suggestions:

> Although the idea of rational numbers (children have probably called them fractions) is not new to children in fourth grade, we would like to first concern ourselves with ideas rather than symbols. In these first experiences an effort has been made to develop the idea that we (1) specify a unit (region, segment, set) (2) partition or separate the unit into congruent or equivalent parts, (3) name the number of parts with which we are concerned. We have deliberately refrained from using the notation $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{8}$, etc. One can use the words one-fourth, one-half, etc.

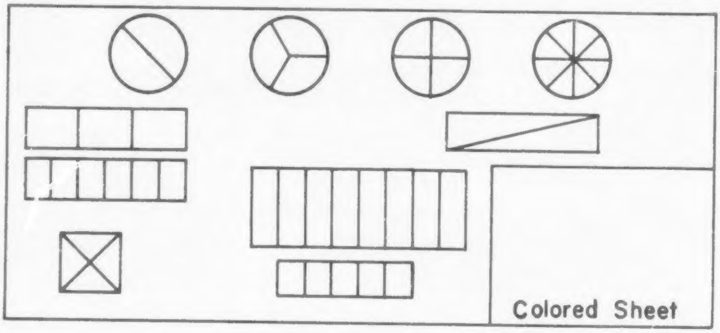
Getting ready for this lesson. Prepare a set of materials so that children may have several models of rectangular regions, and circular regions. An example of such a sheet, that might be prepared and duplicated, is shown below. You may select your own models. Each child can cut out his own models and place them in an envelope until ready for use. (Maybe a homework assignment) Or, you may have such materials already available for each child.

These do not represent actual size. Make each a convenient size that can be used easily by children.



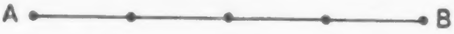
Felt pieces on flannel board may be used by teacher in working with class.

At beginning of class have each child put the parts together on his desk (see illustration below). Also give him a colored card or sheet of paper. This is only for contrast so that he can place the region to be discussed upon the mat for identification.

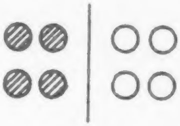


First identify the regions and note into how many parts each has been partitioned. Then ask each child to place on the mat a region which has been separated into three congruent parts. Ask him to identify 1 of these parts, 2 of them, 3 of them, etc. Do similar activities for other regions.

Then use line segments and sets of objects, as

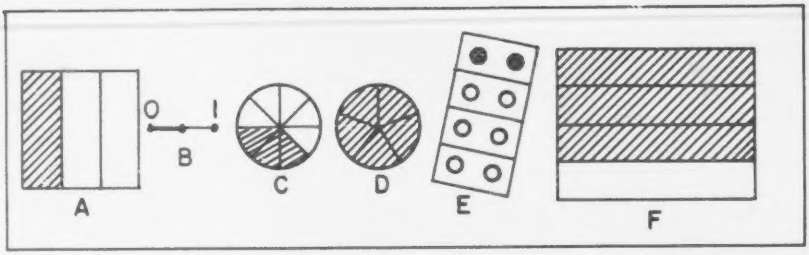


and



(NOTE: These do not represent actual size to be used.)

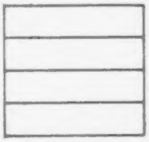
Then, use paper discs where part of the congruent regions of each unit are one color and the others another color, or materials on the flannel board. If nothing else, make drawings on the chalkboard and shade some of the regions, as shown on the following page.

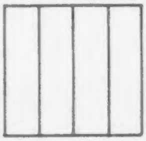


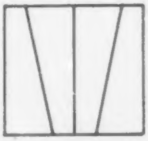
Then, make a chart as the following to record the information. Note that you use a pair of numbers in each instance.

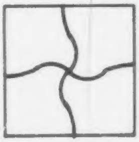
Figure	Number of Congruent Parts in Unit	Number of Parts Shaded
A	3	1
B	2	1
C	8	3
D	5	5
E	4	1
F	4	3

For the last experience, have children find regions (rectangular and circular) separated into 4 congruent parts. Ask if anyone can suggest other ways of separating them into 4 congruent regions. You might have some models available for those who wish to try this on their own and report their findings to the class. Some of their findings might be









Now use the material in pupil text, and Exercise Set 1.

Chapter 10

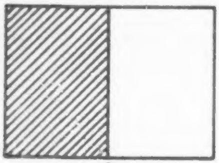
CONCEPT OF RATIONAL NUMBERS

IDEA OF RATIONAL NUMBERS

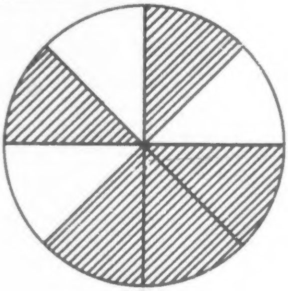
Exploration

Look at each of the figures on this page.

For each figure, choose a pair of numbers at the right which can be used to talk about the number of parts that are shaded and the number of congruent parts into which each unit region, unit segment, or set has been separated.



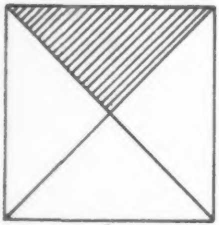
A



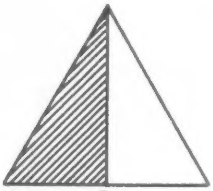
B

Pairs of Numbers

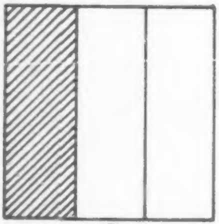
a. 1 and 4
b. 3 and 4
c. 3 and 5
d. 1 and 2
e. 5 and 8
f. 1 and 3
g. 2 and 3
h. 2 and 2
i. 6 and 8
j. 2 and 5



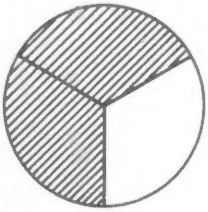
C



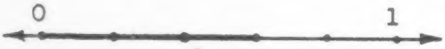
D



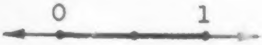
E



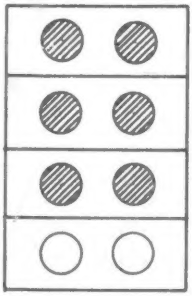
F



G



H



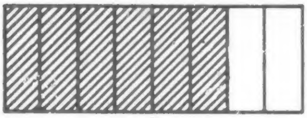
I

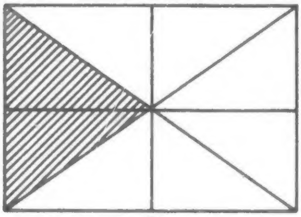
Were you able to find a pair of numbers for each? Did you find these -- A-d; B-e; C-a; D-d; E-f; F-g; G-c; H-h; and I-b?

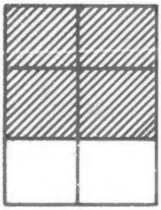
Exercise Set 1

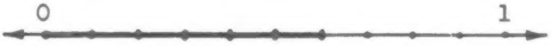
1. Copy the table and complete it, using the figures A, B, C, D, E, and F.

Figure	Parts Shaded	Congruent Parts in Unit
A	6	8
B	2	8
C	4	6
D	6	10
E	2	5
F	6	6

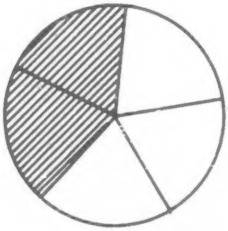

A

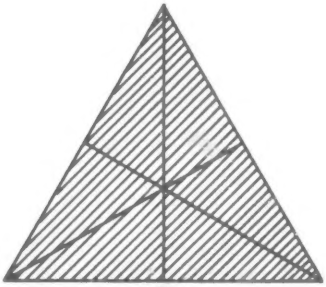

B


C

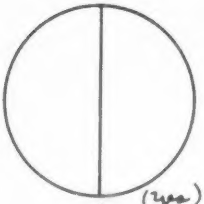


D

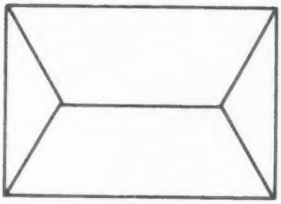

E


F

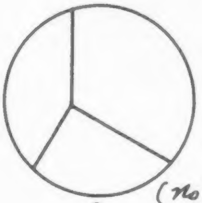
2. Write on your paper the letters from A to G. After each, write Yes if the figure has been partitioned into congruent regions. Write No if the figure has not been partitioned into congruent regions.



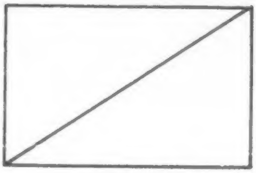
A (yes)



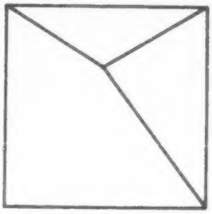
B (no)



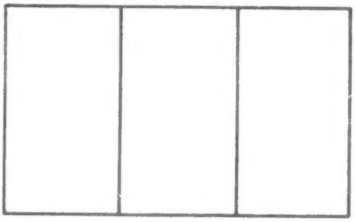
C (no)



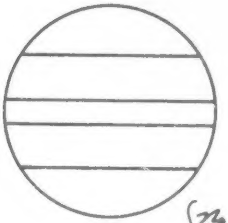
D (yes)



E (no)

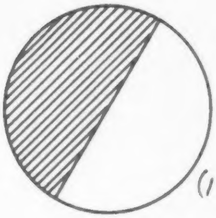


F (yes)



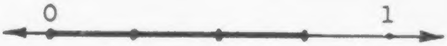
G (no)

3. What pair of numbers can be used to talk about the shaded region in each figure? Remember we will let the first number of the pair tell how many parts are shaded. We will let the second number of the pair tell into how many congruent parts the unit region has been partitioned.

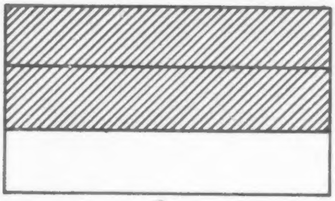


(1 and 2)

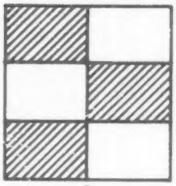
A



B (3 and 4)

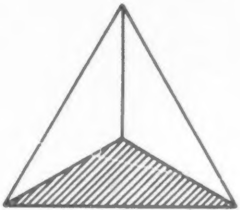


(2 and 3) D

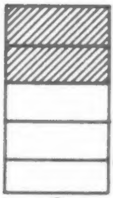


(3 and 6)

C

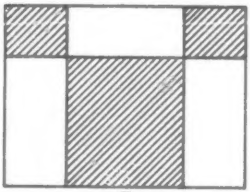


E (1 and 3)

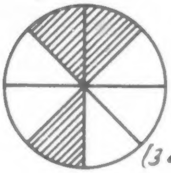


(2 and 5)

G

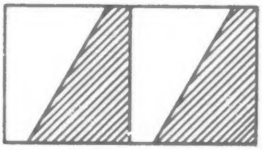


F (none)

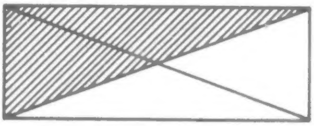


(3 and 8)

H



I (2 and 4)



J (none)

Did you find any figures that had not been partitioned into congruent regions? (yes) Which ones were they? (F, J)

A NEW KIND OF NUMBER

Objective: To learn to use a fraction as a symbol for naming the rational number; and

To help children see that the numerals used in a fraction symbol are associated with the numbers that

(1) identify the number of congruent parts into which a unit region has been partitioned, and

(2) the number of parts with which one is concerned.

Teaching Suggestions:

Recall whole number representations. One can recall the experiences in which we had collections of discrete objects and how these sets of objects suggested to us the whole number.

Review ideas in the last lesson. Then recall the experiences in the preceding lesson by asking what number is suggested by models in the Exploration. You may also use some of the models in exercises.

Then ask if these are different ideas than models which suggest whole numbers.

Rational numbers. Note that these are a new kind of number and that we call them rational numbers.

Names for rational numbers are fractions. Tell how we use the pairs of numbers from the previous lesson to write the name of a rational number. Note that we call these names, *fractions*.

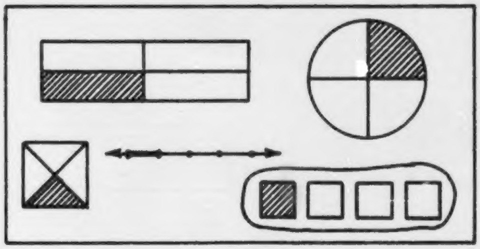
Then you may wish to use the materials in the pupil text as well as previous activities in which you go one step farther to name the number suggested by the shaded region, the region not shaded, etc.

Now use materials in pupil text for *A New Kind of Number*.

A NEW KIND OF NUMBER

Exploration

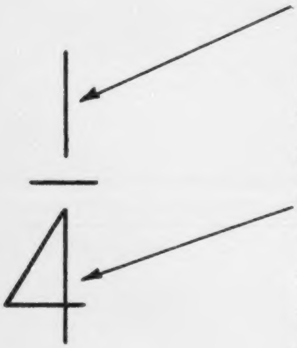
When a region is partitioned into congruent parts and some of these parts are shaded, we use a new kind of number to describe what we see. These new numbers are called rational numbers. $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{3}{8}$ are rational numbers. They are read, "one-half," "one-fourth," and "three-eighths."



Each of these figures at the right suggests the same rational number. The rational number is one-fourth. The symbol, $\frac{1}{4}$, which names the rational number one-fourth is called a fraction. Fractions are written using two numerals. The two numerals are separated by a horizontal bar.

For example:

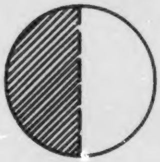
The numerals are 1 and 4.



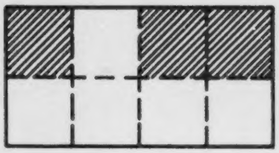
The numeral above the bar tells the number of congruent parts of equivalent subsets described. The number is called the numerator.

The numeral below the bar tells the number of congruent parts into which the set of objects, unit region, or unit segment has been partitioned. The number is called the denominator.

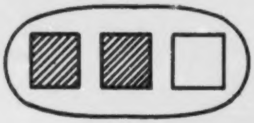
What rational number is suggested by each of these figures below?



A $\left(\frac{1}{2}\right)$

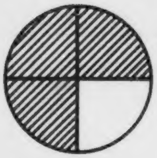


B $\left(\frac{3}{8}\right)$

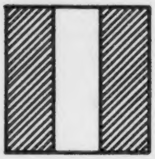


C $\left(\frac{2}{3}\right)$

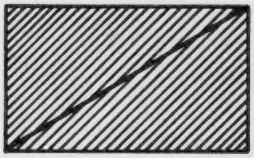
What rational number does each of these figures suggest?



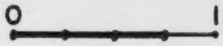
A $\left(\frac{3}{4}\right)$



B $\left(\frac{2}{3}\right)$



C $\left(\frac{2}{2}\right)$



D $\left(\frac{3}{4}\right)$

Figure A suggests the rational number, $\frac{3}{4}$, read three-fourths.

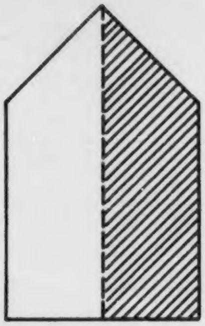
Figure B suggests the rational number, $\frac{2}{3}$, read two-thirds.

Figure C suggests the rational number, $\frac{2}{2}$, read two halves.

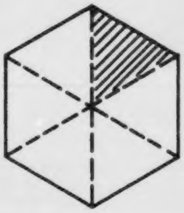
Figure D also suggests the rational number, $\frac{3}{4}$.

Exercise Set 2

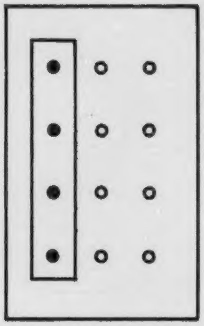
1. For each figure, write a fraction which names the rational number suggested by the shaded region.



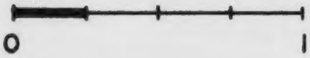
A ($\frac{1}{2}$)



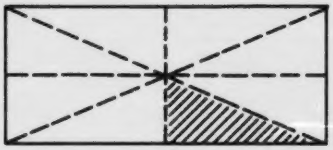
B ($\frac{1}{6}$)



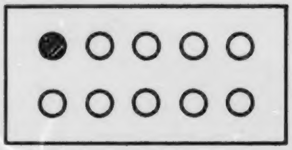
C ($\frac{1}{3}$)



D ($\frac{1}{4}$)



E ($\frac{1}{8}$)



F $\frac{1}{10}$

2. Write as fractions:

a) one-half $\frac{1}{2}$

b) one-third $\frac{1}{3}$

c) one-tenth $\frac{1}{10}$

d) one-eighth $\frac{1}{8}$

e) one-sixth $\frac{1}{6}$

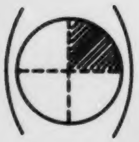
f) one-fourth $\frac{1}{4}$

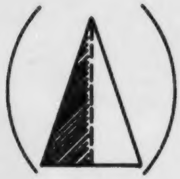
3. Copy the unit square in Figure H at least six times. (Make more copies if you want them.) In how many ways can you separate the unit square to show:

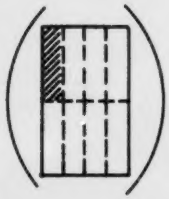
$\frac{1}{2}$? $\frac{1}{4}$? $\frac{1}{8}$? (answers will vary.)

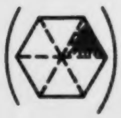
H

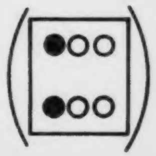
4. Copy and shade the part which is described by the fraction below each figure.

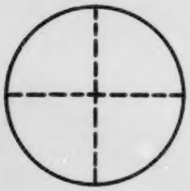






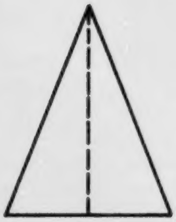






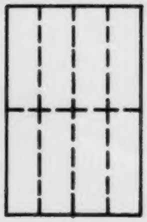
A

$\frac{1}{4}$



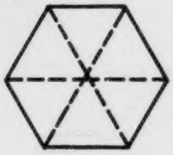
B

$\frac{1}{2}$



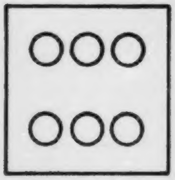
C

$\frac{1}{8}$



D

$\frac{1}{6}$



E

$\frac{1}{3}$

5. Copy and complete this chart.

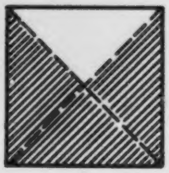
Fraction	Number of Congruent Parts in Unit	Number of Congruent Parts Counted
$\frac{1}{4}$	4	1
$\frac{1}{2}$	2	1
$\frac{1}{10}$	10	1
$\frac{1}{6}$	6	1
$\frac{1}{3}$	3	1
$\frac{1}{8}$	8	1

6. On your paper, make 6 copies of the unit region shown below. Make the unit regions the same size. Then show a picture that suggests each of the rational numbers named in exercise 5. (Answers will vary.)

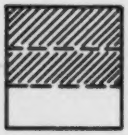
Exercise Set 3

1. Use these figures to complete the chart below.

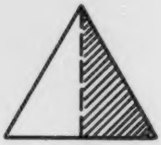
   A has been done for you.



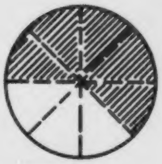
A



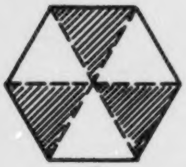
B



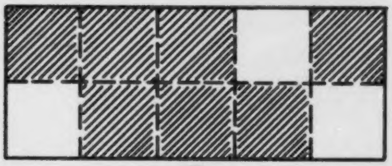
C



D



E



F

Figure	Number of Congruent Parts in Figure	Number of Shaded Parts	Rational Number Suggested by Shaded Part
A	4	3	$\frac{3}{4}$
B	3	2	$\frac{2}{3}$
C	2	1	$\frac{1}{2}$
D	8	5	$\frac{5}{8}$
E	6	3	$\frac{3}{6}$
F	10	7	$\frac{7}{10}$

2. Using figures A, B, C, D, E, and F of exercise 1, write the name of the rational number suggested by the unshaded part of each figure.

A - $\frac{1}{4}$
B - $\frac{1}{3}$
C - $\frac{1}{2}$
D = $\frac{3}{8}$
E - $\frac{3}{6}$
F - $\frac{3}{10}$

3. Use these figures to complete the sentences below.

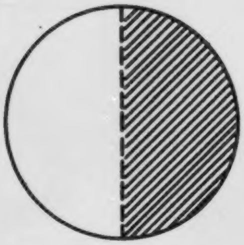


Fig. A

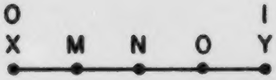


Fig. B

Set C = {1, 2, 4, 6, 8, 10, 12, 14}

Fig. C

a) Figure A has been separated into 2 congruent regions. 1 region has been shaded. The shaded region is best described by the rational number named by the fraction $\frac{1}{2}$.

b) Points M, N, and O separate $\overline{XY}$ into 4 congruent segments. m $\overline{XM}$ = $\frac{1}{4}$.

c) Set C has 8 members. 1 member names an odd number. This member is $\frac{1}{8}$ of all the members of Set C.

4. Study your answers to exercises 1, 2, and 3. Copy and then write "above" or "below" in each blank.

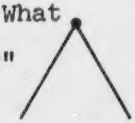
a) The numeral below the bar names the number of congruent parts into which the unit has been separated.

b) The numeral above the bar names the number of congruent parts which are described.

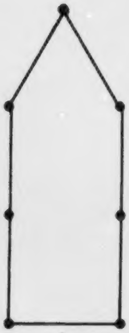
5. Ann watched 3 television programs. Each was $\frac{1}{4}$ of an hour long.

a) How long did Ann watch television? ($\frac{3}{4}$ hour)

b) How much longer would she need to watch TV to make her total time 1 hour? ($\frac{1}{4}$ hour)

6. A figure like the one pictured below was made by laying toothpicks, each the same size, end-to-end. What fractional part of the perimeter is the "roof"  ?

($\frac{2}{7}$)



RATIONAL NUMBERS GREATER THAN ONE

Objective: To associate rational numbers with points on the number line and to extend these numbers beyond those suggested by the unit segments and congruent parts of the unit segment; and

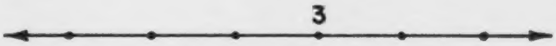
To learn to identify the rational numbers suggested by diagrams when more than one unit region is used.

Materials: Prepare for each child a dittoed sheet of number lines. One should be marked in unit segments. One line each then for $\frac{1}{2}$ units, $\frac{1}{4}$ units, $\frac{1}{8}$ units, $\frac{1}{5}$ units, $\frac{1}{3}$ units, $\frac{1}{6}$ units, $\frac{1}{10}$ units, and $\frac{1}{12}$ units. (This is for the purpose of helping children name the number associated with the points which will partition the unit segment into the given congruent parts as indicated.)

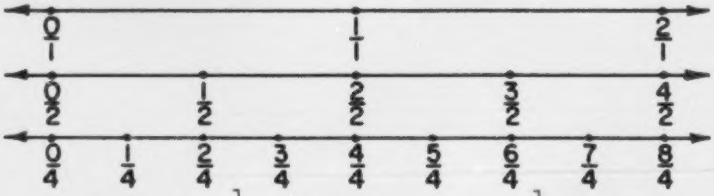
Teaching Suggestions:

Recall with the children that we have learned to recognize what rational numbers are suggested by unit parts of regions, line segments and unit sets of objects.

Recall that we have known how to associate points on the number line with the whole numbers. Then draw a line on the chalkboard and perhaps locate a point with which 3 can be associated. Let the class suggest the numbers for the other points.



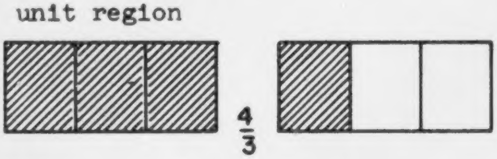
Then with sets of number lines suggested in Materials, first have them associate rational numbers with the points in the line marked in unit segments, then with the line in half-unit segments, etc., as



Count by $\frac{1}{2}$'s beginning from $\frac{1}{2}$, by 2-fourths beginning with one-fourth, etc. Count not only forward but backward.

Then ask them to find one and one-half unit on the number line (expect them to select the one-half unit line to be used.) Ask what fraction names the number. Do many other similar activities, that is, find the rational number named by the fraction $\frac{8}{4}$, using the one-eighth unit number line, etc.

Then using either materials on the flannel board or drawing on the chalkboard, ask them to name the rational number associated with each, as

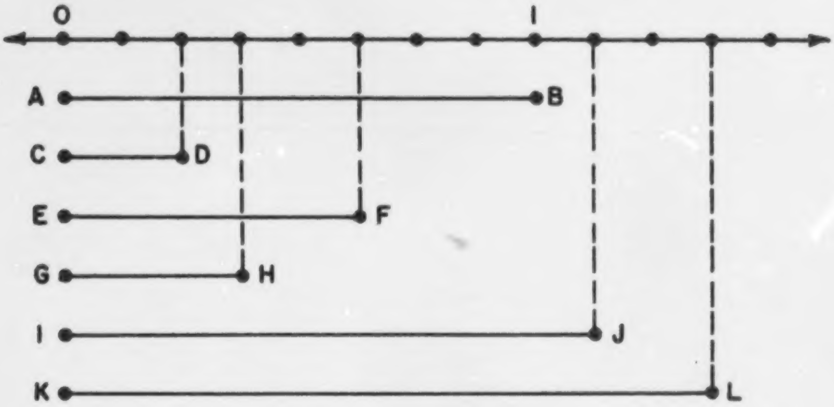


(Note. If it would seem convenient and also helpful to name a rational number using the mixed form at this time, do so. Note with your pupils that we can name a number as $\frac{3}{2}$ in two ways--either using the fraction or naming the number of unit segments and the parts of unit segments, as 1 and $\frac{1}{2}$. Also, you may wish to show how we can write 1 and $\frac{1}{2}$ in a shorter way, that is, $1\frac{1}{2}$. We are not interested in changing from fractions to mixed forms or vice versa, using only symbols. All of this should be done using the number line, or other models as rectangular and circular regions. You may want to use some of materials that come later in this chapter here. They are included in the section, "A New Kind of Name."

RATIONAL NUMBERS GREATER THAN ONE

Exploration

In the picture below, the line segment AB is 1 unit long.



1. (a) On the number line the unit segment is separable into __8__ congruent segments.

(b) Use a fraction: Each small segment is $\frac{1}{8}$ of the unit segment.

(c) The measure of $\overline{AB}$ is 1. The measure of $\overline{AB}$ is also $\frac{8}{8}$. (Use a fraction.)

(d) Is $\frac{8}{8}$ the measure of line segment AB? (yes)

(e) Is $\frac{2}{8}$ the measure of line segment CD? (yes)

(f) Is $\frac{5}{8}$ the measure of line segment EF? (yes)

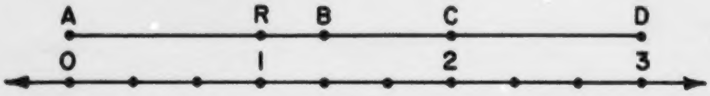
(g) Is $\frac{3}{8}$ the measure of line segment GH? (yes)

(h) Is $\frac{9}{8}$ the measure of line segment IJ? (yes)

(i) Is $\frac{11}{8}$ the measure of line segment KL? (yes)

P528

2. Each unit segment of the number line below has been separated into 3 congruent segments. $\overline{AR}$ is the same length as the unit segment.

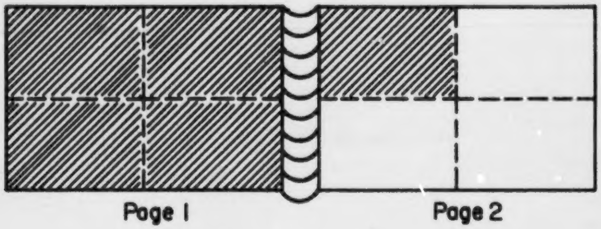


Use this number line to answer the questions.

(a) What fraction names the measure of $\overline{AR}$? $\left(\frac{3}{3}\right)$

(b) What fraction names the measure of $\overline{AB}$? $\overline{AC}$? $\overline{AD}$?
$\left(\frac{4}{3}\right)$ $\left(\frac{6}{3}\right)$ $\left(\frac{9}{3}\right)$

3. Bill has a photograph album. Each page is separated into 4 congruent parts. On each page he can place 4 pictures.



If Bill pastes 5 pictures in his album, he will cover $\frac{4}{4}$ of one page and $\frac{1}{4}$ of another page. What rational number describes the number of pages covered? (one and one-fourth)

Fractions like $\frac{2}{8}$, $\frac{5}{8}$, $\frac{3}{8}$, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{4}{5}$ tell us that the measure of a segment or a region is less than 1.

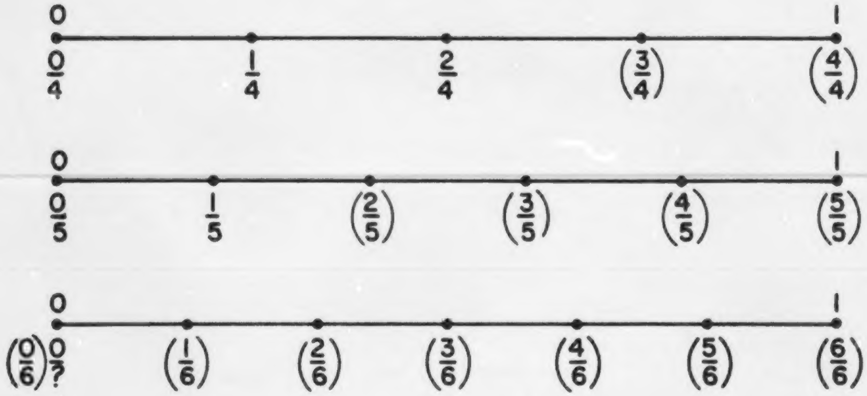
Fractions like $\frac{8}{8}$, $\frac{2}{2}$, $\frac{3}{3}$, $\frac{4}{4}$, $\frac{5}{5}$, $\frac{6}{6}$ tell us that the measure of a segment or region is exactly 1.

Fractions like $\frac{9}{8}$, $\frac{11}{8}$, $\frac{3}{2}$, $\frac{4}{3}$, $\frac{5}{4}$, $\frac{5}{3}$, $\frac{5}{2}$ tell us that the measure of a segment or region is greater than 1.

P529

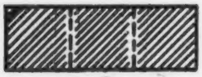
Exercise Set 4

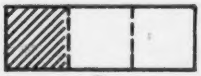
1. Copy the unit segments below. The dots separate each unit segment into smaller, congruent segments. Label each dot correctly.

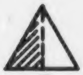

0
1
0/4
1/4
2/4
(3/4)
(4/4)
0
1
0/5
1/5
(2/5)
(3/5)
(4/5)
(5/5)
0
1
(0/6)
0/?
(1/6)
(2/6)
(3/6)
(4/6)
(5/6)
(6/6)

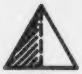
Each of the figures below represents a unit region or unit segment.

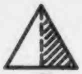
2. Study these diagrams. Then answer the questions on the next page.

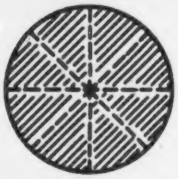

A

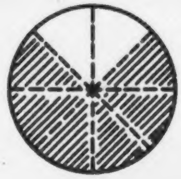

B

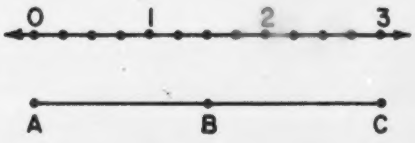

C


D


E


F


G



H

P530

a) How many thirds are there in A? (3)

How many thirds are there in B? (3)

How many thirds are shown in A and B together? (6)

What rational fraction is suggested by the shaded region of A and B together? $\left(\frac{4}{3}\right)$

What rational number is suggested by the unshaded region of A and B together? $\left(\frac{2}{3}\right)$

---

b) What rational number is suggested by the shaded region in C? $\left(\frac{1}{2}\right)$ in D? $\left(\frac{1}{2}\right)$ in E? $\left(\frac{1}{2}\right)$

What rational number is suggested by the unshaded region in C? $\left(\frac{1}{2}\right)$ in D? $\left(\frac{1}{2}\right)$ in E? $\left(\frac{1}{2}\right)$

What rational number best describes the shaded regions in C, D, and E altogether? $\left(\frac{3}{2}\right)$

What rational number best describes the unshaded regions in C, D, and E altogether? $\left(\frac{3}{2}\right)$

c) What rational number is suggested by the shaded region of F and G together? $\left(\frac{14}{8}\right)$

What rational number is suggested by the unshaded region of F and G together? $\left(\frac{2}{8}\right)$

d) In Figure H, what rational number is the measure of $\overline{AB}$? $\left(\frac{6}{4}\right)$ of $\overline{AC}$? $\left(\frac{12}{4}\right)$

3. For each figure, write the fraction that names the rational number suggested by the shaded part.

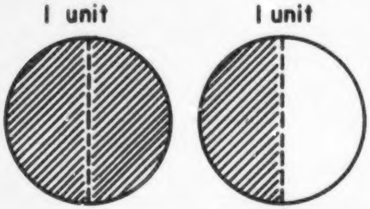

I unit
I unit

FIG. A $\left(\frac{3}{2}\right)$

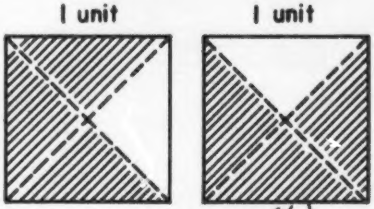

I unit
I unit

FIG. B $\left(\frac{6}{4}\right)$

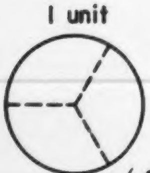


FIG. C $\left(\frac{0}{3}\right)$

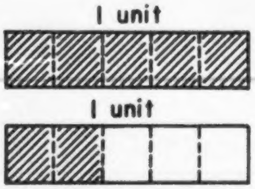

I unit
I unit

FIG. D $\left(\frac{7}{5}\right)$

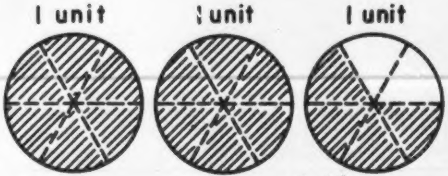
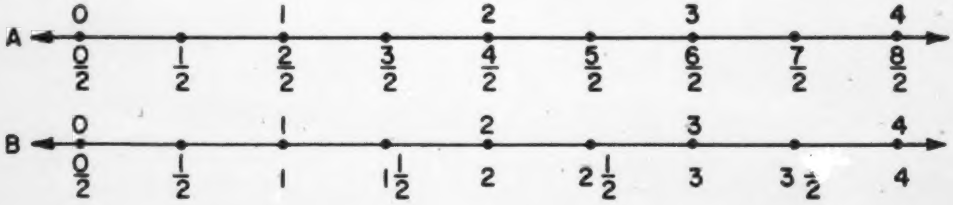

I unit
I unit
I unit

FIG. E $\left(\frac{16}{6}\right)$

4. Using these number lines, complete the sentences below.


A
0
1
2
3
4
0/2
1/2
2/2
3/2
4/2
5/2
6/2
7/2
8/2
B
0
1
2
3
4
0/2
1/2
1
1½
2
2½
3
3½
4

a) 1 one and 1 half = $\frac{3}{2}$ or $1\frac{1}{2}$

b) 4 = $\frac{8}{2}$

c) 3 ones and 1 half = $\frac{7}{2}$ or $3\frac{1}{2}$

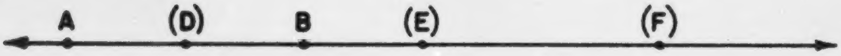
d) 2 ones and 1 half = $\frac{5}{2}$ or $2\frac{1}{2}$

e) $\frac{3}{2}$ = 1 one and 1 half

f) 2 = $\frac{4}{2}$

g) $\frac{2}{2}$ = 1

5. Copy the line segment shown below on your paper.



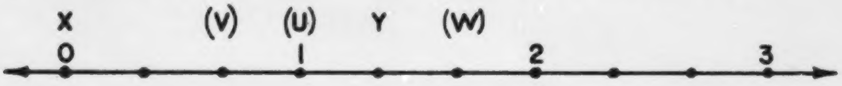
$\overline{AB}$ is a unit segment.

a) Mark a point D so that $\overline{AD}$ is $\frac{1}{2}$ unit long.

b) Mark a point E so that $\overline{AE}$ is $\frac{3}{2}$ units long.

c) Mark a point F so that $\overline{AF}$ is $\frac{5}{2}$ units long.

6. Copy the line segment below. Notice each unit segment has been separated into 3 congruent segments.



Using a certain unit, the measure of $\overline{XY}$ is $\frac{4}{3}$.
Mark new points U, V, and W so that

a) $\overline{XU}$ is 1 unit long.

b) $\overline{XV}$ is $\frac{2}{3}$ unit long.

c) $\overline{XW}$ is $\frac{5}{3}$ units long.

7. Mark is 4 feet tall. What number gives his height in yards? (one and one-third)

8. Ellen watched 5 television programs. How many hours did she watch TV if each program was:

a) $\frac{1}{4}$ of an hour long? (one hour and one quarter hour)

b) $\frac{1}{2}$ of an hour long? (2 hours and $\frac{1}{2}$ hour)

DIFFERENT NAMES FOR THE SAME RATIONAL NUMBER

Objective: To help children see that the same rational number has many names; and

To help children learn a way to recognize the simplest name.

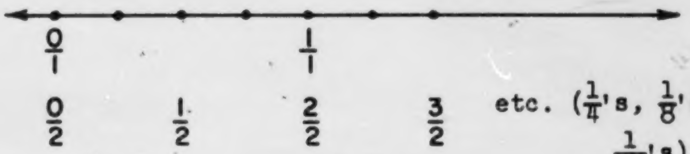
Materials: A fraction chart, and/or circular regions showing the unit, $\frac{1}{2}$'s, $\frac{1}{4}$'s, $\frac{1}{3}$'s, $\frac{1}{5}$'s, $\frac{1}{6}$'s, $\frac{1}{8}$'s, $\frac{1}{10}$'s, $\frac{1}{12}$'s

Several number lines,
one for the "family of $\frac{1}{2}$'s, $\frac{1}{4}$'s, $\frac{1}{8}$'s, $\frac{1}{12}$'s;"
one for the "family of $\frac{1}{3}$'s, $\frac{1}{6}$'s, $\frac{1}{12}$'s;" and
one for the "family of $\frac{1}{2}$'s, $\frac{1}{5}$'s, and $\frac{1}{10}$'s."

Teaching Suggestions:

Using the fraction chart, have children find those parts of regions which show the same part of the unit. For example, $\frac{1}{2}$, $\frac{2}{4}$, $\frac{4}{8}$, $\frac{6}{12}$, $\frac{3}{6}$, and $\frac{5}{10}$ all show the same part of the unit region.
Explore others, as $\frac{1}{4}$, $\frac{3}{4}$, $\frac{1}{3}$, etc.
Note with them that these show the same part.
We express this idea by saying $\frac{1}{2} = \frac{2}{4} = \frac{4}{8}$, etc.
We can give a set of fractions which would do the same thing. Write the fractions and then include others which might be members of the set, although you do not have models for them.

Now use the number lines. However, instead of writing the fractions below for each line, write each set of fractions, letting them be associated with the points of the same line, that is, as indicated above.



etc. ($\frac{1}{4}$'s, $\frac{1}{8}$'s, $\frac{1}{12}$'s)

Observe that the same point has several fractions below it. We may write these as sets of fractions as

$$\frac{1}{2}, \frac{2}{4}, \frac{4}{8}, \frac{6}{12}, \text{ etc.}$$

Again, you may wish to ask if anyone could give another fraction that would be a member of the set. If there is hesitation, you might suggest that one number of the pair is 20 and write 20 below the bar. Ask them what would go above. Or you could give a first number of the pair and ask that they give the other.

You also might ask if anyone has an idea of how they could keep on finding other fractions.

Continue with the others where you note that there are several names for the same number.

Make a special observation of those which are names for the whole numbers.

Raise the question: If we have several names, how do we choose the one that we should use? Note that the one where pair of numbers is least is often called the basic fraction. That is,

$\frac{1}{2}$ would be our choice from the set

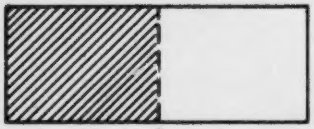
$$\left\{\frac{1}{2}, \frac{2}{4}, \frac{4}{8}, \cdots\right\}$$

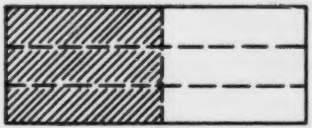
Now use materials in the pupil text.

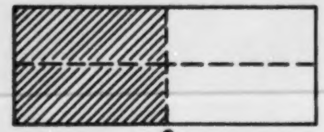
DIFFERENT NAMES FOR THE SAME NUMBER

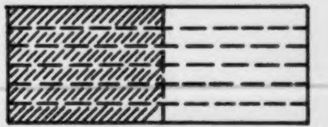
Exploration

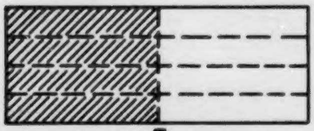
1. The pictures of unit regions below suggest some ways of thinking of one-half.

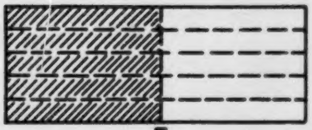

A


B


C


D


E


F

In A, what fraction names the measure of the shaded region? $(\frac{1}{2})$

In B, what fraction names the measure of the shaded region? $(\frac{3}{6})$

In C, what fraction names the measure of the shaded region? $(\frac{2}{4})$

In D, what fraction names the measure of the shaded region? $(\frac{6}{12})$

In E, what fraction names the measure of the shaded region? $(\frac{4}{8})$

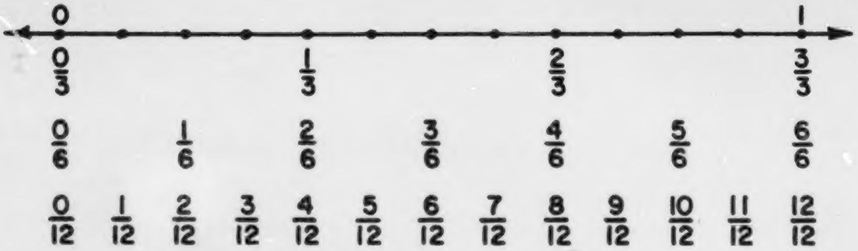
In F, what fraction names the measure of the shaded region? $\frac{5}{10}$

$\frac{1}{2}$, $\frac{3}{6}$, $\frac{2}{4}$, $\frac{6}{12}$, $\frac{4}{8}$, and $\frac{5}{10}$ are all ways of naming the rational number $\frac{1}{2}$.

We can write: $\frac{1}{2} = \frac{3}{6} = \frac{4}{8} = \frac{5}{10}$.

What are some other fractions that name this same number? $(\frac{2}{4}, \frac{6}{12}, \frac{7}{14},$ etc.$)$

We say that $\frac{1}{2}$ is the simplest name, or simplest form, for this rational number. Can you tell why? (1 and 2 are the pair with the smallest numbers.)


0
1
0/3
1/3
2/3
3/3
0/6
1/6
2/6
3/6
4/6
5/6
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0/12
1/12
2/12
3/12
4/12
5/12
6/12
7/12
8/12
9/12
10/12
11/12
12/12

2. Make true statements by writing a fraction in each blank. Use the number line above to help you.

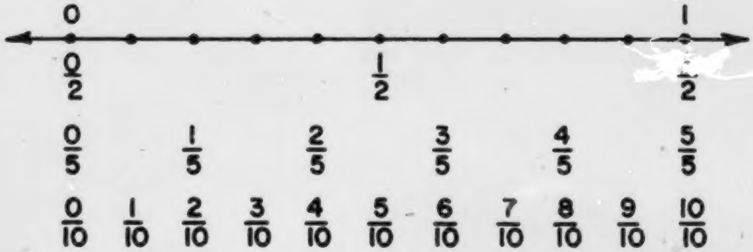
a. $\frac{1}{6} = \underline{\frac{2}{12}}$

b. $\frac{10}{12} = \underline{\frac{5}{6}}$

c. $\frac{1}{3} = \underline{\frac{2}{6}} = \underline{\frac{4}{12}}$

d. $\frac{4}{6} = \underline{\frac{2}{3}} = \underline{\frac{8}{12}}$

e. $1 = \underline{\frac{3}{3}} = \underline{\frac{6}{6}} = \underline{\frac{12}{12}}$


0
1
0/2
1/2
2/2
0/5
1/5
2/5
3/5
4/5
5/5
0/10
1/10
2/10
3/10
4/10
5/10
6/10
7/10
8/10
9/10
10/10

3. Use the number line above to help you write the missing numerator or denominator.

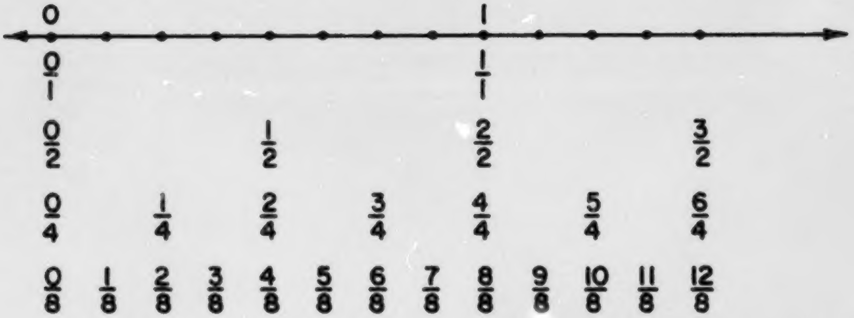
a. $\frac{1}{2} = \frac{n}{10}$ $(n=5)$

b. $\frac{2}{10} = \frac{1}{n}$ $(n=5)$

c. $\frac{4}{5} = \frac{n}{10}$ $(n=8)$

d. $1 = \frac{10}{n} = \frac{s}{5}$ $\left(\begin{array}{l} n=10 \\ s=5 \end{array}\right)$

4. Using one number line, we can show many different names for a rational number.



We see that some fractions are names for the same rational number.

What other fractions are names for the rational number $\frac{1}{2}$? $\left(\frac{2}{4}, \frac{4}{8}\right)$

What other fraction is a name for the rational number $\frac{1}{4}$? $\left(\frac{2}{8}\right)$

What other fraction is a name for the rational number $\frac{3}{4}$? $\left(\frac{6}{8}\right)$

Can you find other fractions that name the same rational number on this line? $\left(\frac{2}{2}, \frac{6}{4}, \frac{12}{8}\right.$, etc.)

One rational number may be named by many fractions.

The rational number $\frac{1}{4}$ may be named by: $\{\frac{1}{4}, \frac{2}{8}, \frac{3}{12}, \frac{6}{24}, \ldots\}$

The rational number $\frac{2}{3}$ may be named by: $\{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots\}$

The rational number $\frac{2}{5}$ may be named by: $\{\frac{2}{5}, \frac{4}{10}, \frac{6}{15}, \frac{8}{20}, \ldots\}$

The rational number $\frac{1}{10}$ may be named by: $\{\frac{1}{10}, \frac{2}{20}, \frac{3}{30}, \frac{4}{40}, \ldots\}$

Can you think of other fractions which would name each of these numbers above? $\left(\frac{7}{28}, \frac{10}{15}, \frac{10}{25}, \frac{5}{50}\right.$, etc.)

Many fractions can be used to name the same whole numbers.

For example, 1 may be indicated by

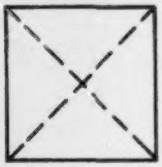
$\frac{2}{2}, \frac{3}{3}, \frac{4}{4}, \frac{5}{5}, \frac{6}{6},$ and so on.

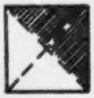
Can you name three other fractions that belong to this set? $\left(\frac{7}{7}, \frac{8}{8}, \frac{9}{9}\right.$, etc.)

Exercise Set 5

Copy each of these figures.

1.

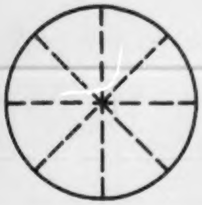


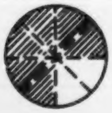


Color $\frac{2}{4}$ of this figure.

$\frac{2}{4}$ is another name for $\frac{1}{2}$.

2.

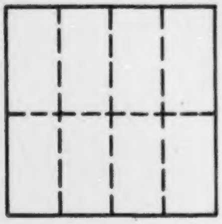


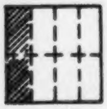


Color $\frac{6}{8}$ of this figure.

$\frac{6}{8}$ is another name for $\frac{3}{4}$.

3.

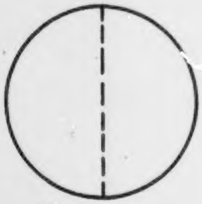


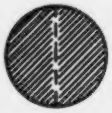


Color $\frac{2}{8}$ of this figure.

$\frac{2}{8}$ is another name for $\frac{1}{4}$.

4.

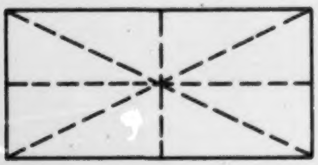




Color $\frac{2}{2}$ of this figure.

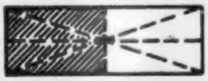
$\frac{2}{2}$ is another name for 1.

5.

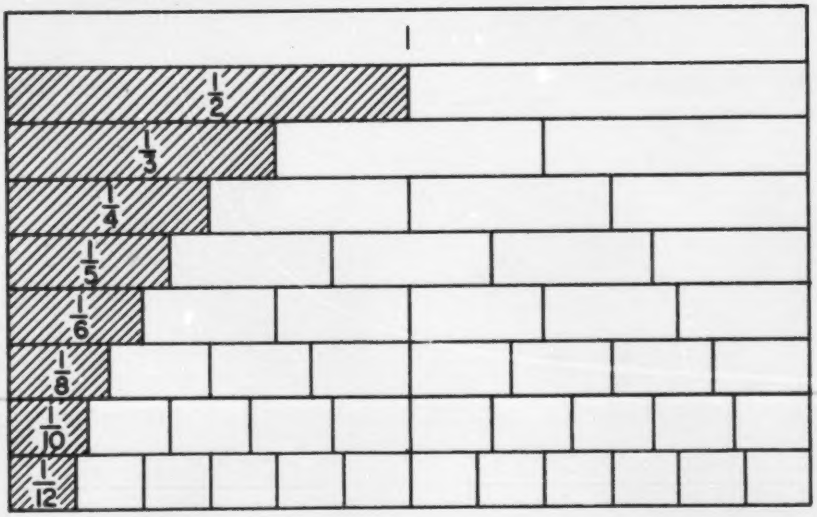


Color $\frac{4}{8}$ of this figure.

$\frac{4}{8}$ is another name for $\frac{1}{2}$.



Using this chart, write as many names as you can for



a) $\frac{1}{2}$ $\left(\frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}, \frac{6}{12}\right)$

b) $\frac{1}{3}$ $\left(\frac{2}{6}, \frac{4}{12}\right)$

c) $\frac{1}{4}$ $\left(\frac{2}{8}, \frac{3}{12}\right)$

d) $\frac{1}{5}$ $\left(\frac{2}{10}\right)$

e) $\frac{4}{6}$ $\left(\frac{2}{3}, \frac{8}{12}\right)$

f) $\frac{6}{8}$ $\left(\frac{3}{4}, \frac{9}{12}\right)$

7. Write at least three other fractions which name each of the following rational numbers. If you can write more than three, do so.

a) $\frac{1}{4}$ $\left(\frac{2}{8}, \frac{3}{12}, \frac{4}{16}\right)$

b) $\frac{2}{5}$ $\left(\frac{4}{10}, \frac{6}{15}, \frac{8}{20}\right)$

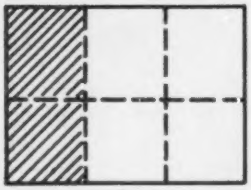
c) $\frac{5}{2}$ $\left(\frac{10}{4}, \frac{15}{6}, \frac{20}{8}\right)$

d) $\frac{2}{3}$ $\left(\frac{4}{6}, \frac{6}{9}, \frac{8}{12}\right)$

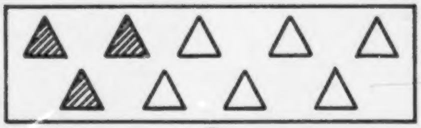
e) $\frac{3}{4}$ $\left(\frac{6}{8}, \frac{9}{12}, \frac{12}{16}\right)$

(Many other fractions are acceptable.)

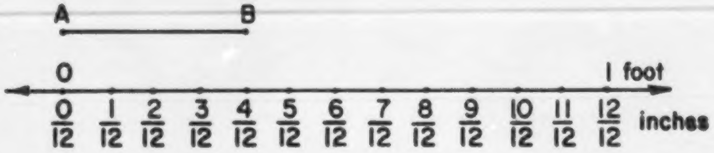
8. The diagrams below suggest three other names for $\frac{1}{3}$. What are they? $\left(\frac{2}{6}, \frac{3}{9}, \frac{4}{12}\right)$



A

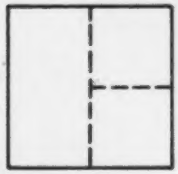


B


A
B
0
1 foot
$\frac{0}{12}$ $\frac{1}{12}$ $\frac{2}{12}$ $\frac{3}{12}$ $\frac{4}{12}$ $\frac{5}{12}$ $\frac{6}{12}$ $\frac{7}{12}$ $\frac{8}{12}$ $\frac{9}{12}$ $\frac{10}{12}$ $\frac{11}{12}$ $\frac{12}{12}$
inches

C

9. Draw 5 boxes like the ones below. Separate each box to show the mathematical sentence written below. The first one is done for you.



$\frac{1}{2} = \frac{2}{4}$

$\frac{4}{8} = \frac{1}{2}$

$\frac{1}{4} = \frac{2}{8}$

$\frac{6}{8} = \frac{3}{4}$

$\frac{2}{6} = \frac{1}{3}$

10. Complete:

a) $1 = \frac{2}{2}$

b) $\frac{2}{6} = \frac{1}{3}$

c) $\frac{1}{2} = \frac{2}{4}$

d) $\frac{1}{4} = \frac{2}{8}$

e) $\frac{4}{8} = \frac{1}{2}$

f) $\frac{8}{8} = \frac{4}{4}$

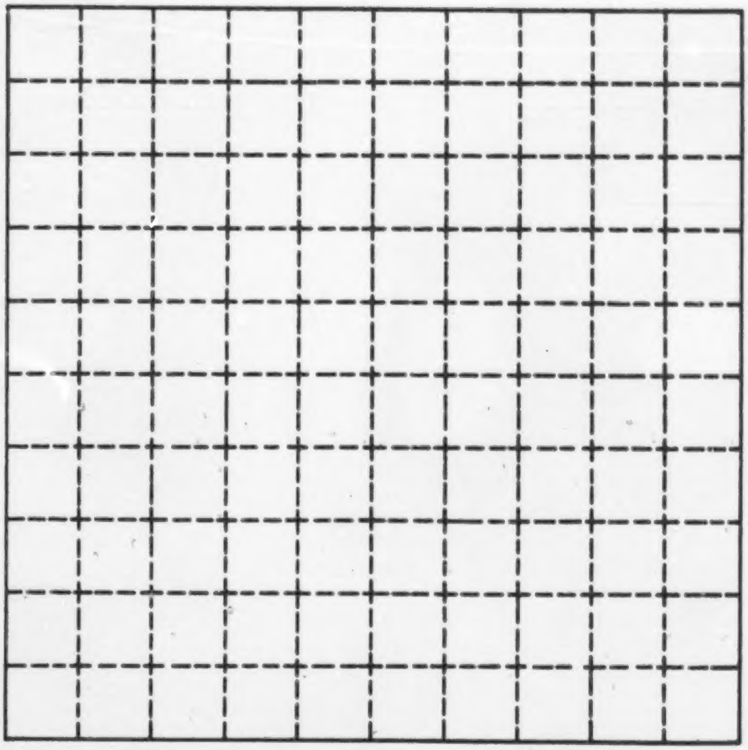
g) $\frac{1}{3} = \frac{3}{9}$

h) $\frac{2}{4} = \frac{4}{8}$

i) $\frac{6}{8} = \frac{3}{4}$

j) $\frac{1}{3} = \frac{4}{12}$

11.



P540

The unit square shown on the preceding page has been separated into 100 congruent square regions.

a) Each small square region is what part of the unit square region? $\left(\frac{1}{100}\right)$

b) Each small square region is what part of 1 row or 1 column of square regions? $\left(\frac{1}{10}\right)$

c) Each row or each column of square regions is what part of the unit square region? $\left(\frac{1}{10}\right)$

d) $\frac{4}{10} = \frac{?^{40}}{100}$

e) $\frac{7}{10} = \frac{?^{70}}{100}$

f) $\frac{90}{100} = \frac{?^{9}}{10}$

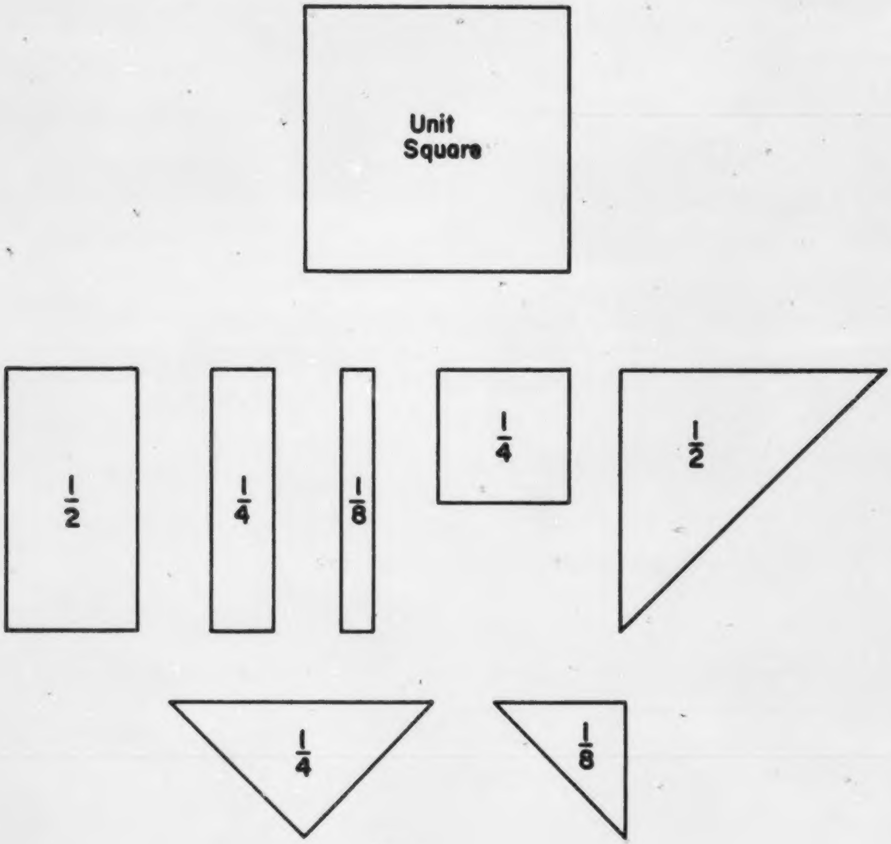
g) $\frac{30}{100} = \frac{?^{3}}{10}$

h. How many small square regions should you color if you are to color $\frac{47}{100}$ of the unit square region? (47)

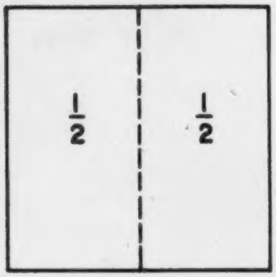
$\frac{83}{100}$? (83) $\frac{100}{100}$? (100) $\frac{1}{10}$? (10) $\frac{7}{10}$? (70) and $\frac{10}{10}$? (100)

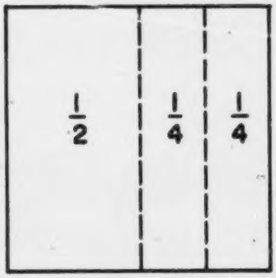
P541

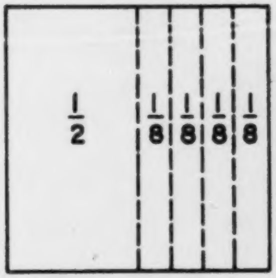
Puzzle. In how many different ways can you cover the unit square using the fractional pieces shown? Each piece may be used more than once. You may wish to trace, cut out, and make several copies of each model region before you work your puzzle.


Unit Square
$\frac{1}{2}$
$\frac{1}{4}$
$\frac{1}{8}$
$\frac{1}{4}$
$\frac{1}{2}$
$\frac{1}{4}$
$\frac{1}{8}$

Here are a few solutions to the puzzle on page 541. How many more did you discover?






1/2
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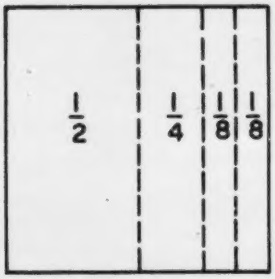

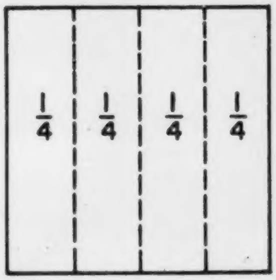

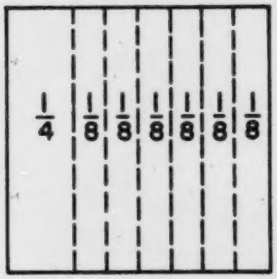
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1/8
1/8
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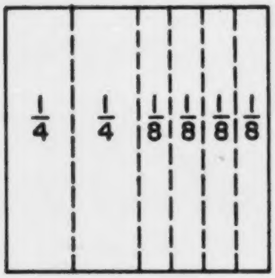
1/8


1/4
1/4
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1/8
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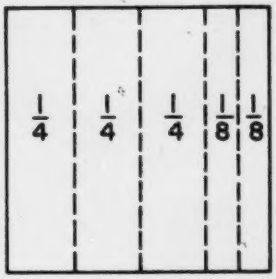
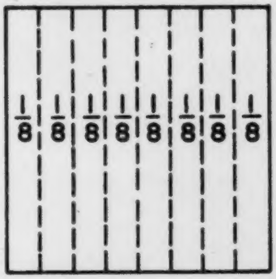
1/8


1/4
1/4
1/4
1/8
1/8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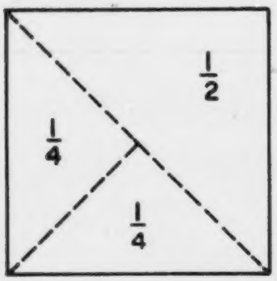
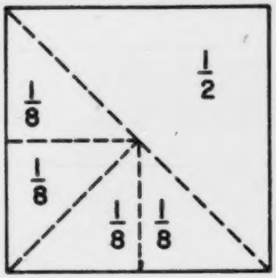
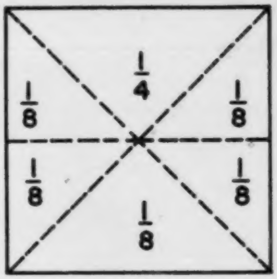
1/8




1/8
1/2
1/8
1/8
1/8


1/4
1/8
1/8
1/8
1/8
1/8

ORDERING THE RATIONAL NUMBERS

Objective: To help children learn that if they know one rational number they cannot name the next one;

To order a given set of rational numbers from least to greatest and from greatest to least; and

To express a "less than" and "greater than" relationship between two rational numbers.

Materials: Those which have been used in previous lessons.

Teaching Suggestions:

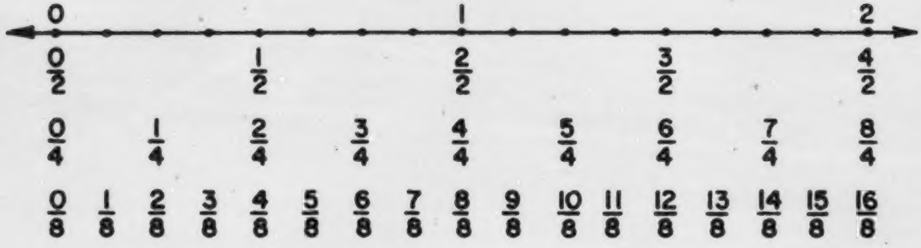
Background for this lesson should have grown out of the preceding lessons. Children already are probably aware that $\frac{1}{3} > \frac{1}{4}$, etc. Using the number line or congruent parts of a given region, ask some questions which would lead them to order a given set of rational numbers from least to greatest.

For example, given the set $\frac{1}{2}$, $\frac{3}{4}$, $\frac{1}{3}$, the ordered set would be $\frac{1}{3}$, $\frac{1}{2}$, $\frac{3}{4}$. Ask them to give the next whole number after 5; after 123, etc.

Follow this by asking them to give the next rational number after $\frac{1}{2}$. Should they give $\frac{2}{2}$, then ask them if, say $\frac{3}{4}$, is between $\frac{1}{2}$ and $\frac{2}{2}$. The series of questions you ask here should focus on the idea that given one rational number, the next rational number cannot be given.

ORDERING THE RATIONAL NUMBERS

Exploration



Look at the number line.

Is $\frac{1}{2}$ to the right of $\frac{1}{4}$? (yes) Is $\frac{1}{2} > \frac{1}{4}$? (yes)

Is $\frac{3}{4}$ to the right of $\frac{3}{8}$? (yes) Is $\frac{3}{4} > \frac{3}{8}$? (yes)

Is $\frac{5}{4}$ to the right of $\frac{4}{2}$? (no) Is $\frac{5}{4} > \frac{4}{2}$? (no)

Is 0 to the left of $\frac{1}{4}$? (yes) Is $0 < \frac{1}{4}$? (yes)

Is $\frac{2}{4}$ to the left of $\frac{4}{2}$? (yes) Is $\frac{2}{4} < \frac{4}{2}$? (yes)

Is $\frac{5}{8}$ to the left of $\frac{6}{8}$? (yes) Is $\frac{5}{8} < \frac{6}{8}$? (yes)

It is easy to see that $\frac{1}{4}$, $\frac{3}{4}$, $\frac{6}{4}$, and $\frac{8}{4}$ are ordered from least to greatest.

Are $\frac{1}{2}$, $\frac{5}{8}$, $\frac{5}{4}$, and $\frac{4}{2}$ ordered from the least to greatest? (yes)

It would be easier to decide if we used other fractions for these numbers.

Using other names for these same numbers, we can write them as $\frac{4}{8}$, $\frac{5}{8}$, $\frac{10}{8}$, and $\frac{16}{8}$.

Now we see the numbers are named in order from least to greatest.

As you move to the right along a number line, the rational numbers become greater. As you move to the left, they become less.

Exercise Set 6

1. Use this chart and the symbols $>$ and $<$ to complete the sentences below.

<table>
<tr><td colspan="8">$\frac{1}{2}$</td><td colspan="8">$\frac{1}{2}$</td></tr>
<tr><td colspan="4">$\frac{1}{4}$</td><td colspan="4">$\frac{1}{4}$</td><td colspan="4">$\frac{1}{4}$</td><td colspan="4">$\frac{1}{4}$</td></tr>
<tr><td colspan="2">$\frac{1}{8}$</td><td colspan="2">$\frac{1}{8}$</td><td colspan="2">$\frac{1}{8}$</td><td colspan="2">$\frac{1}{8}$</td><td colspan="2">$\frac{1}{8}$</td><td colspan="2">$\frac{1}{8}$</td><td colspan="2">$\frac{1}{8}$</td><td colspan="2">$\frac{1}{8}$</td></tr>
<tr><td>$\frac{1}{16}$</td><td>$\frac{1}{16}$</td><td>$\frac{1}{16}$</td><td>$\frac{1}{16}$</td><td>$\frac{1}{16}$</td><td>$\frac{1}{16}$</td><td>$\frac{1}{16}$</td><td>$\frac{1}{16}$</td><td>$\frac{1}{16}$</td><td>$\frac{1}{16}$</td><td>$\frac{1}{16}$</td><td>$\frac{1}{16}$</td><td>$\frac{1}{16}$</td><td>$\frac{1}{16}$</td><td>$\frac{1}{16}$</td><td>$\frac{1}{16}$</td></tr>
</table>

a) $\frac{1}{2} \underline{\ >\ } \frac{1}{4}$

b) $\frac{1}{8} \underline{\ <\ } \frac{1}{4}$

c) $\frac{3}{8} \underline{\ >\ } \frac{1}{4}$

d) $\frac{1}{2} \underline{\ >\ } \frac{3}{8}$

e) $\frac{3}{4} \underline{\ >\ } \frac{5}{8}$

f) $\frac{3}{8} \underline{\ <\ } \frac{2}{4}$

2. Write the correct answer. The fraction chart above may be used, if needed.

a) Which number is less: $\frac{17}{8}$ or $\frac{16}{8}$? $\left(\frac{16}{8}\right)$

Which is farther to the left on the number line? $\left(\frac{16}{8}\right)$

b) Which number is less: $\frac{10}{8}$ or $\frac{12}{8}$? $\left(\frac{10}{8}\right)$

Which is farther to the left on the number line? $\left(\frac{10}{8}\right)$

c) Which number is less: $\frac{17}{8}$ or $\frac{15}{8}$? $\left(\frac{15}{8}\right)$

Which is farther to the left on the number line? $\left(\frac{15}{8}\right)$

d) Which number is less: $\frac{11}{4}$ or $\frac{4}{2}$? $\left(\frac{4}{2}\right)$

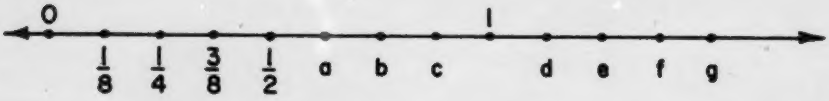
Which is farther to the left on the number line? $\left(\frac{4}{2}\right)$

3. Arrange members of each set in order from least to greatest. Make diagrams if you need them.

$A = \left\{\frac{7}{2}, \frac{3}{2}, \frac{11}{2}, \frac{13}{2}, \frac{5}{2}\right\}$ $\left(\frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{11}{2}, \frac{13}{2}\right)$

$B = \left\{\frac{7}{4}, \frac{3}{4}, 2, \frac{9}{4}, \frac{11}{4}\right\}$ $\left(\frac{3}{4}, \frac{7}{4}, 2, \frac{9}{4}, \frac{11}{4}\right)$

4. Associate a rational number with points a, b, c, d, e, f, and g in the diagram below.



$a = \frac{5}{8}$

$b = \frac{3}{4}$ or $\frac{6}{8}$

$c = \frac{7}{8}$

$d = \frac{9}{8}$

$e = \frac{5}{4}$

$f = \frac{11}{8}$

$g = \frac{6}{4}$

5. List in order the numbers used in counting by two-thirds from $\frac{2}{3}$ to 4. $\left(\frac{2}{3}, \frac{4}{3}, \frac{6}{3}, \frac{8}{3}, \frac{10}{3}, \frac{12}{3}\right)$

6. List in order the numbers used in counting by three-halves from $\frac{3}{2}$ to 9. $\left(\frac{3}{2}, \frac{6}{2}, \frac{9}{2}, \frac{12}{2}, \frac{15}{2}, \frac{18}{2}\right)$

7. Write two other names for each of the following numbers.

a) $\frac{12}{8}$ $\left(\frac{6}{4}, \frac{3}{2}\right)$

c) $\frac{10}{4}$ $\left(\frac{5}{2}, \frac{20}{8}\right)$

b) $\frac{5}{2}$ $\left(\frac{10}{4}, \frac{15}{6}\right)$

d) 3 $\left(\frac{6}{2}, \frac{9}{3}\right)$

8. Copy and complete by writing the symbol > or < in each box.

a) $\frac{1}{4} \underline{<} \frac{1}{2}$ d) $1 \underline{>} \frac{1}{2}$

b) $\frac{1}{2} \underline{>} \frac{1}{8}$ e) $\frac{1}{4} \underline{>} \frac{1}{8}$

c) $\frac{1}{10} \underline{<} 1$ f) $\frac{1}{6} \underline{<} \frac{1}{3}$

9. Rearrange these numbers in order from least to greatest.

a) $\frac{5}{8}, \frac{3}{8}, \frac{3}{4}, \frac{1}{8}, \frac{1}{4}$ $(\frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{5}{8}, \frac{3}{4})$

b) $\frac{1}{3}, \frac{1}{5}, \frac{1}{2}, \frac{1}{6}, \frac{1}{4}$ $(\frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2})$

c) $\frac{2}{3}, \frac{5}{6}, \frac{1}{6}, \frac{1}{3}$ $(\frac{1}{6}, \frac{1}{3}, \frac{2}{3}, \frac{5}{6})$

d) $\frac{3}{8}, \frac{1}{2}, \frac{3}{4}, \frac{5}{5}$ $(\frac{3}{8}, \frac{1}{2}, \frac{3}{4}, \frac{5}{5})$

10. Arrange in order the numbers in each set below. Begin with the greatest.

$A = \left\{\frac{2}{4}, \frac{3}{4}, \frac{1}{4}\right\}$ $\{\frac{3}{4}, \frac{2}{4}, \frac{1}{4}\}$

$B = \left\{\frac{1}{4}, \frac{1}{2}, \frac{1}{8}\right\}$ $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}\}$

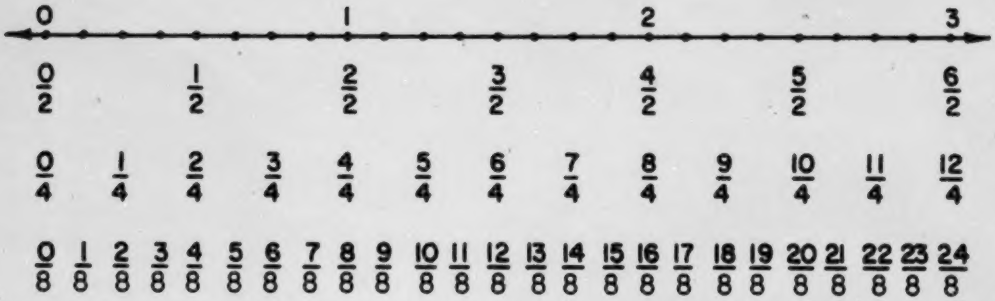
$C = \left\{\frac{1}{2}, \frac{3}{8}, \frac{3}{4}\right\}$ $\{\frac{3}{4}, \frac{1}{2}, \frac{3}{8})$

11. Arrange these numbers from least to greatest.

$\frac{1}{2}, \frac{1}{10}, \frac{1}{4}, \frac{1}{8}, \frac{1}{3}, \frac{1}{6}.$ $(\frac{1}{10}, \frac{1}{8}, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2})$

Exercise Set 7

Supplementary Exercises


0
1
2
3
0/2 1/2 2/2 3/2 4/2 5/2 6/2
0/4 1/4 2/4 3/4 4/4 5/4 6/4 7/4 8/4 9/4 10/4 11/4 12/4
0/8 1/8 2/8 3/8 4/8 5/8 6/8 7/8 8/8 9/8 10/8 11/8 12/8 13/8 14/8 15/8 16/8 17/8 18/8 19/8 20/8 21/8 22/8 23/8 24/8

1. Copy and write >, <, or = in each blank to make a true sentence. The number line above will help you.

a) $\frac{5}{4} \underline{\quad > \quad} \frac{1}{2}$

b) $\frac{8}{4} \underline{\quad = \quad} 2$

c) $3 \underline{\quad = \quad} \frac{6}{2}$

d) $\frac{3}{2} \underline{\quad > \quad} \frac{3}{4}$

e) $\frac{2}{2} \underline{\quad = \quad} 1$

f) $\frac{3}{4} \underline{\quad > \quad} \frac{5}{8}$

g) $\frac{7}{4} \underline{\quad > \quad} \frac{11}{8}$

h) $\frac{18}{8} \underline{\quad > \quad} \frac{8}{4}$

2. Which fraction of each pair below will be farther to the right on the number line?

a) $\frac{19}{8}$ or $\frac{17}{8}$ $\left(\frac{19}{8}\right)$

b) $\frac{11}{5}$ or $\frac{5}{4}$ $\left(\frac{11}{5}\right)$

c) $\frac{5}{2}$ or $\frac{18}{8}$ $\left(\frac{5}{2}\right)$

d) $\frac{10}{2}$ or $\frac{5}{2}$ $\left(\frac{10}{2}\right)$

e) $\frac{14}{8}$ or $\frac{6}{4}$ $\left(\frac{14}{8}\right)$

f) $\frac{11}{4}$ or $\frac{4}{2}$ $\left(\frac{11}{4}\right)$

g) $\frac{5}{4}$ or $\frac{3}{2}$ $\left(\frac{3}{2}\right)$

h) $\frac{1}{2}$ or $\frac{1}{4}$ $\left(\frac{1}{2}\right)$

3. Rearrange each set. Put members in order from least to greatest.

$A = \left\{\frac{7}{2}, \frac{3}{2}, \frac{11}{2}, \frac{5}{2}\right\}$ $\left(\left\{\frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{11}{2}\right\}\right)$

$B = \left\{\frac{1}{4}, \frac{1}{2}, \frac{1}{3}, \frac{1}{8}\right\}$ $\left(\left\{\frac{1}{8}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}\right\}\right)$

4. Copy and fill in each blank with the symbol $>$, $<$, or $=$.

a) $\frac{3}{3}\ \underline{\ >\ }\ \frac{1}{3}$

b) $2\ \underline{\ >\ }\ \frac{4}{3}$

c) $\frac{4}{4}\ \underline{\ =\ }\ 1$

d) $\frac{4}{5}\ \underline{\ >\ }\ \frac{2}{5}$

e) $1\ \underline{\ <\ }\ 2$

f) $\frac{3}{3}\ \underline{\ =\ }\ \frac{2}{2}$

g) $2\ \underline{\ >\ }\ \frac{7}{4}$

h) $\frac{1}{2}\ \underline{\ <\ }\ 1$

5. Look at exercise 4. Which fraction in each pair labels a point farther to the right on the number line?

(a. $\frac{3}{3}$ c. same point e. 2 g. 2
b. 2 d. $\frac{4}{5}$ f. same point h. 1)

A NEW KIND OF NAME

Objective: To show that the mixed form is convenient to use when the rational number is greater than 1.

Materials: The number line and congruent regions.

Teaching Suggestions:

Go back to review the lesson in which we considered rational numbers greater than one, noting how we used a fraction to name a rational number greater than 1.

Then make some statements like these:

Tom rode his bicycle $\frac{3}{2}$ miles.

Mary's mother used $\frac{10}{6}$ pies for her guests at dinner.

I would like someone to get a piece of string for me which is $\frac{4}{3}$ yards long.

Get children's reactions to such statements. Ask what might be a more convenient way to make these statements. That is, if $\frac{3}{2}$, $\frac{10}{6}$, and $\frac{4}{3}$ could be expressed as a number of whole units and parts of units.

Here children suggest:

Tom rode his bicycle $1\frac{1}{2}$ miles.

Mary's mother used $1\frac{4}{6}$ pies for her dinner guests.

I would like someone to get a piece of string for me which is $1\frac{1}{3}$ yards long.

Then ask: If we know the fraction for a rational number, how can we express it using whole units and parts of units?

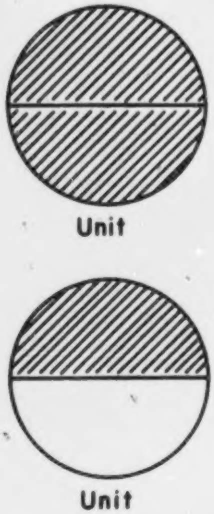
$1\frac{1}{2}$, $1\frac{4}{6}$, etc., are what we call mixed forms. Then ask, "If Tom rode his bicycle $\frac{3}{2}$ miles, did he ride as far as one mile? More than one mile? As far as two miles?" etc. "Is a piece of string $\frac{4}{3}$ yards long as long as one yard? Longer than one yard? As long as two yards?" etc. This will help children recognize the relationship between rational numbers named by fractions and whole numbers.

P549

A NEW KIND OF NAME

These pictures help us **think** about the numbers, $\frac{3}{2}$ and $\frac{11}{4}$.

A.



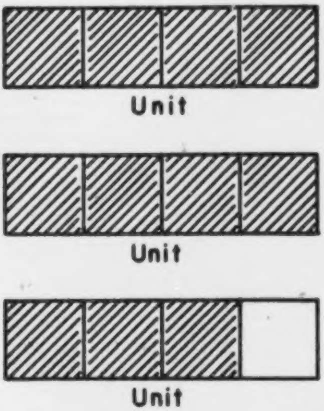
$\frac{3}{2} = \frac{2}{2}$ and $\frac{1}{2}$

$\frac{3}{2} = 1$ one and 1 half

or,

$\frac{3}{2} = 1\frac{1}{2}$

B.



$\frac{11}{4} = \frac{4}{4}$ and $\frac{4}{4}$ and $\frac{3}{4}$

$\frac{11}{4} = 1$ one and 1 one
and 3 fourths

$\frac{11}{4} = 2$ ones and
3 fourths

or,

$\frac{11}{4} = 2\frac{3}{4}$

Another way of naming $\frac{3}{2}$ is $1\frac{1}{2}$

Another way of naming $\frac{11}{4}$ is $2\frac{3}{4}$

We call $1\frac{1}{2}$ and $2\frac{3}{4}$ mixed forms.

Rational numbers named by fractions like $\frac{1}{2}$, $\frac{3}{4}$, $\frac{2}{3}$, and $\frac{7}{8}$ tell us that the measure of a region, segment, or set is less than 1.

Rational numbers named by fractions like $\frac{2}{2}$, $\frac{8}{8}$, $\frac{4}{4}$, and $\frac{3}{3}$ tell us that the measure of a region, segment, or set is equal to 1.

Rational numbers named by fractions like $\frac{7}{4}$, $\frac{3}{2}$, and $\frac{5}{3}$ tell us that the measure of a region, segment, or set is greater than 1.

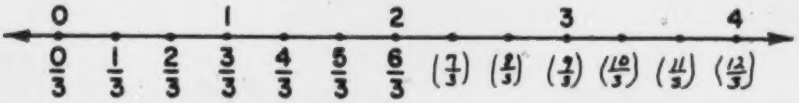
Other names for 1 are $\frac{4}{4}$, $\frac{2}{2}$, and $\frac{3}{3}$.

Since this is true, $\frac{7}{4}$, $\frac{3}{2}$, and $\frac{5}{3}$ may be renamed $1\frac{3}{4}$, $1\frac{1}{2}$, and $1\frac{2}{3}$.

$1\frac{3}{4}$, $1\frac{1}{2}$, and $1\frac{2}{3}$ are read, "one and three-fourths," "one and one-half," and "one and two-thirds". Fractions written in this way are said to be in mixed form.

Exercise Set 8

1. Copy and finish the number line below. Then use it to complete the mathematical sentences so that each will be a true sentence.


0
1
2
3
4
$\frac{0}{3}$
$\frac{1}{3}$
$\frac{2}{3}$
$\frac{3}{3}$
$\frac{4}{3}$
$\frac{5}{3}$
$\frac{6}{3}$
$(\frac{7}{3})$
$(\frac{8}{3})$
$(\frac{9}{3})$
$(\frac{10}{3})$
$(\frac{11}{3})$
$(\frac{12}{3})$

a) $\frac{4}{3} = \frac{3}{3}$ and $\frac{1}{3}$

$\frac{4}{3} = 1\frac{1}{3}$

b) $\frac{6}{3}$ is $\frac{3}{3}$ and $\frac{3}{3}$

$\frac{6}{3} = 2$

c) $\frac{11}{3} = 3$ ones and $\frac{2}{3}$

$\frac{11}{3} = 3\frac{2}{3}$

d) $2\frac{2}{3} = 2$ and $\frac{2}{3}$

$2\frac{2}{3} = \frac{6}{3}$ and $\frac{2}{3}$

$2\frac{2}{3} = \frac{8}{3}$

e) $4 = \frac{12}{3}$

f) $3\frac{1}{3} = \frac{9}{3}$ and $\frac{1}{3}$

$3\frac{1}{3} = \frac{10}{3}$

2. Arrange the numbers in each of the following sets in order from least to greatest. Use diagrams if you need them.

$A = \left\{\frac{3}{4}, 0, \frac{7}{4}, 2, \frac{9}{4}, 1\right\}$ $(A = \{0, \frac{3}{4}, 1, \frac{7}{4}, 2, \frac{9}{4}\})$

$B = \left\{\frac{5}{3}, 1, \frac{10}{3}, 4, \frac{7}{3}, 2, \frac{2}{3}, 3\right\}$

$(B = \{\frac{2}{3}, 1, \frac{5}{3}, 2, \frac{7}{3}, 3, \frac{10}{3}, 4\})$

3. Peter has 13 blocks to walk to school. Each block is $\frac{1}{10}$ mile long. How many miles does he have to walk to school?

$(\frac{13}{10}$ or $1\frac{3}{10}$ miles$)$

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4. A pound of butter is usually divided into four bars of the same size. Vicky found 7 bars of butter in her refrigerator. How many pounds of butter were in the refrigerator? ($\frac{7}{4}$ or $1\frac{3}{4}$ pounds)

5. Can you do these without any help? Try some of them.

Write the mixed form for each of these numbers.

a) $\frac{5}{4} = 1\frac{1}{4}$
b) $\frac{6}{2} = 3$
c) $\frac{8}{5} = 1\frac{3}{5}$
d) $\frac{9}{2} = 4\frac{1}{2}$
e) $\frac{12}{5} = 2\frac{2}{5}$
f) $\frac{7}{3} = 2\frac{1}{3}$

6. Which is greater? Write the name of the greater number in each pair. You may use a number line to help you decide.

a) $\frac{5}{3}$ or $1\frac{1}{2}$ ($\frac{5}{3}$)
b) $2\frac{3}{4}$ or $\frac{10}{4}$ ($2\frac{3}{4}$)
c) $\frac{12}{3}$ or $3\frac{7}{8}$ ($\frac{12}{3}$)
d) 6 or $\frac{21}{3}$ ($\frac{21}{3}$)
e) $\frac{8}{7}$ or $1\frac{1}{8}$ ($\frac{8}{7}$)
f) $3\frac{1}{6}$ or $\frac{5}{2}$ ($3\frac{1}{6}$)
g) $4\frac{2}{3}$ or $\frac{31}{10}$ ($4\frac{2}{3}$)
h) $1\frac{3}{4}$ or $\frac{8}{4}$ ($\frac{8}{4}$)

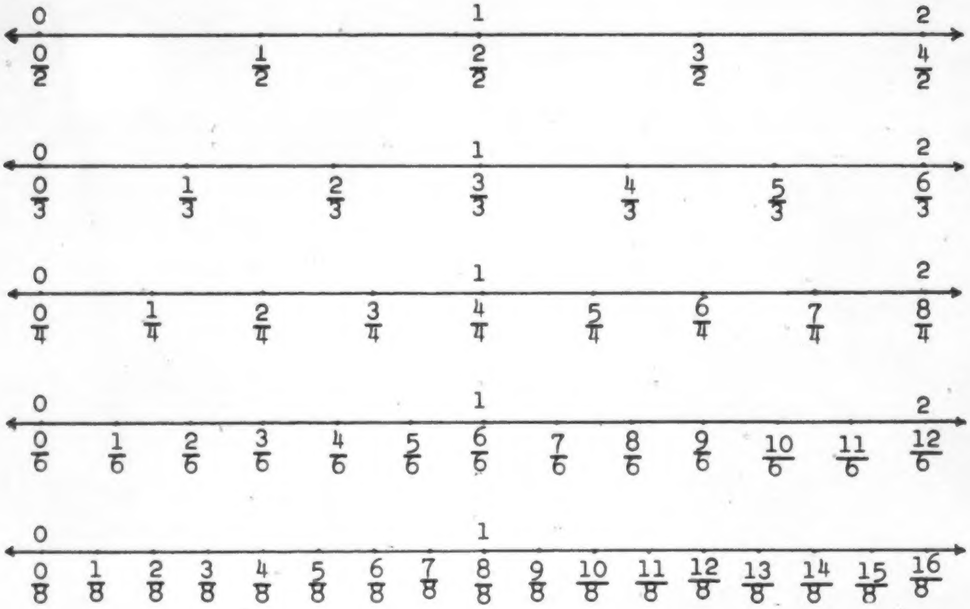
7. Copy and complete. Use diagrams if you need them.

a) $1\frac{2}{5} = \frac{7}{5}$
b) $2\frac{1}{3} = \frac{7}{3}$
c) $2\frac{3}{4} = \frac{11}{4}$
d) $3\frac{1}{2} = \frac{7}{2}$
e) $1\frac{5}{6} = \frac{11}{6}$
f) $2\frac{3}{8} = \frac{19}{8}$

8. Between what two whole numbers on the number line would the following fractions be?

a) $5\frac{1}{3}$ (5 and 6)
b) $2\frac{3}{4}$ (2 and 3)
c) $7\frac{1}{2}$ (7 and 8)
d) $6\frac{9}{10}$ (6 and 7)

Exercise Set 9


0 1 2
0/2 1/2 2/2 3/2 4/2
0 1 2
0/3 1/3 2/3 3/3 4/3 5/3 6/3
0 1 2
0/4 1/4 2/4 3/4 4/4 5/4 6/4 7/4 8/4
0 1 2
0/6 1/6 2/6 3/6 4/6 5/6 6/6 7/6 8/6 9/6 10/6 11/6 12/6
0 1
0/8 1/8 2/8 3/8 4/8 5/8 6/8 7/8 8/8 9/8 10/8 11/8 12/8 13/8 14/8 15/8 16/8

1. Use the number lines above. Copy the following mathematical sentences. Write the symbol $>$ or $<$ in each blank to make the sentence true.

a) $\frac{10}{8} \underline{\ <\ } \frac{12}{8}$

b) $2 \underline{\ >\ } \frac{3}{4}$

c) $\frac{5}{6} \underline{\ <\ } 1$

d) $\frac{3}{2} \underline{\ >\ } \frac{5}{4}$

e) $\frac{3}{6} \underline{\ <\ } \frac{9}{8}$

f) $\frac{8}{8} \underline{\ <\ } \frac{4}{3}$

g) $\frac{17}{4} \underline{\ >\ } \frac{11}{8}$

h) $\frac{4}{3} \underline{\ >\ } \frac{7}{6}$

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2. Starting at zero, list in order the numbers used in

a) counting by one-half to $\frac{4}{2}$ $\left(\frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \frac{4}{2}\right)$

b) counting by two-thirds to $\frac{6}{3}$ $\left(\frac{2}{3}, \frac{4}{3}, \frac{6}{3}\right)$

c) counting by three-eighths to $\frac{15}{8}$ $\left(\frac{3}{8}, \frac{6}{8}, \frac{9}{8}, \frac{12}{8}, \frac{15}{8}\right)$

3. Write 2 other names for each of the following.

a) $\frac{1}{2} = \underline{\frac{2}{4}}, \underline{\frac{4}{8}}$

b) $1 = \underline{\frac{2}{2}}, \underline{\frac{3}{3}}$ (and many others)

c) $1\frac{1}{4} = \underline{\frac{5}{4}}, \underline{\frac{10}{8}}$

d) $2\frac{1}{2} = \underline{\frac{5}{2}}, \underline{\frac{10}{4}}$

4. Match each rational number in Column 1 with a fraction that names the same number from Column 2.

	Column 1		Column 2
a)	$\frac{1}{2}$ (h)	f)	$\frac{8}{4}$
b)	$\frac{1}{4}$ (j)	g)	$\frac{6}{4}$
c)	2 (f)	h)	$\frac{2}{4}$
d)	$1\frac{1}{2}$ (g)	i)	$\frac{2}{2}$
e)	1 (i)	j)	$\frac{2}{8}$

USING RATIONAL NUMBERS

Objectives: To recall that sets of objects and their equivalent subsets have been used to suggest rational numbers; and

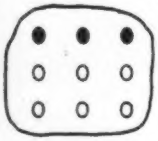
To learn how we can use a rational number to express a relationship between the number of members in a set and the number of members in a subset.

Materials: Collections of objects that can be arranged into sets and subsets.

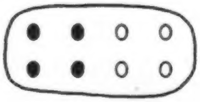
Teaching Suggestions:

Use some sets of objects. Designate a particular subset within a set. Ask what rational number this suggests.

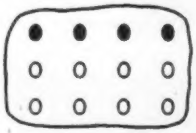
For example,



$\frac{3}{9}$ or $\frac{1}{3}$



$\frac{4}{8}$ or $\frac{1}{2}$



$\frac{4}{12}$ or $\frac{1}{3}$

Then tell how we can use the rational number to speak about the number of members of the set and the number of members in a subset. Again illustrate:

We can say:

3 objects are $\frac{1}{3}$ of 9 objects.

4 objects are $\frac{4}{8}$ (or $\frac{1}{2}$) of 8 objects.

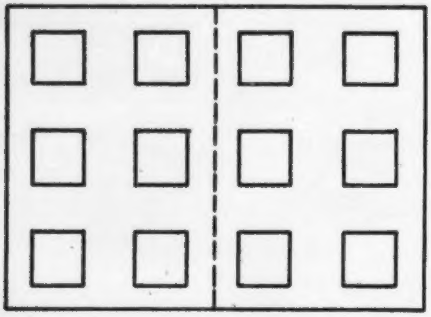
4 objects are $\frac{4}{12}$ (or $\frac{1}{3}$) of 12 objects, etc.

Use the materials in the pupil text.

USING RATIONAL NUMBERS

Exploration

Below are pictures of sets of 12 objects.



Set A

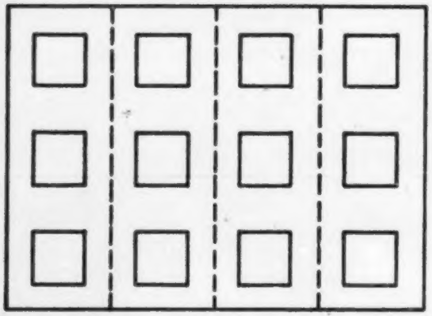
Dotted lines separate the picture of Set A into 2 subsets.

How many objects are there in 1 subset? (6)

How many objects are there in 2 subsets? (12)

Is $\frac{1}{2}$ of 12 objects equal to 6 objects? (yes)

Is $\frac{2}{2}$ of 12 objects equal to 12 objects? (yes)



Set B

Set B has been separated into 4 subsets.

How many objects are in each subset? (3)

$\frac{1}{4}$ of 12 = 3

$\frac{2}{4}$ of 12 = 6

$\frac{3}{4}$ of 12 = 9

$\frac{4}{4}$ of 12 = 12

Is $\frac{1}{2}$ of 12 = $\frac{2}{4}$ of 12? (yes)

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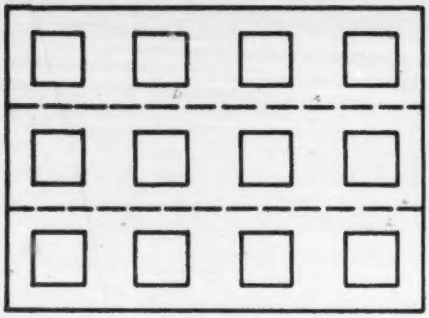


Set C

Dotted lines separate Set C into

3 subsets.

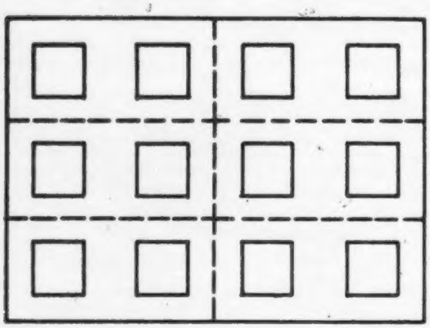
What is $\frac{1}{3}$ of 12? (4)

What is $\frac{2}{3}$ of 12? (8)

What is $\frac{3}{3}$ of 12? (12)

Set D has been separated into

6 subsets.



Set D

$2 = \frac{1}{6}$ of 12.

$4 = \frac{2}{6}$ of 12.

$6 = \frac{3}{6}$ of 12.

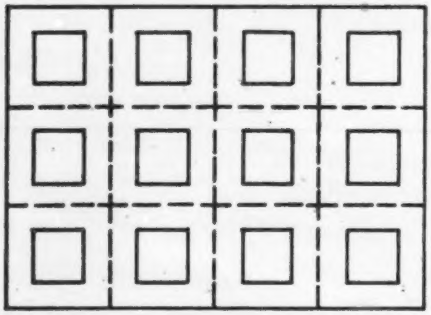
$8 = \frac{4}{6}$ of 12.

$10 = \frac{5}{6}$ of 12.

$12 = \frac{6}{6}$ of 12.

Each subset in E shows

$\frac{1}{12}$ of 12.



Set E

$\frac{3}{12}$ of 12 = 3

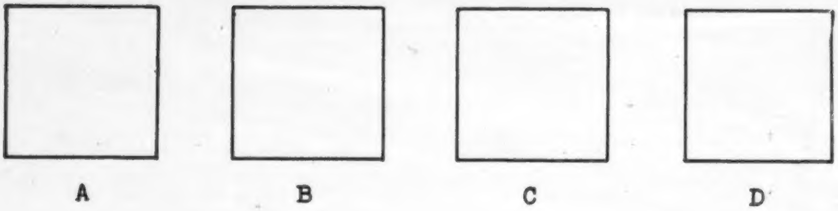
$\frac{4}{12}$ of 12 = 4

$\frac{6}{12}$ of 12 = 6

$\frac{8}{12}$ of 12 = 8

$\frac{9}{12}$ of 12 = 9

Exercise Set 10



1. A, B, C, and D are unit square regions. Copy them on your paper. Separate each one into four equal regions.

a) Color $\frac{1}{4}$ of A red.

b) Color $\frac{2}{4}$ of B blue.

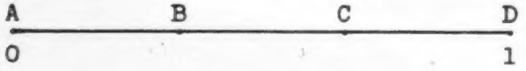
c) Color $\frac{3}{4}$ of C green.

d) Color $\frac{4}{4}$ of D green.

e) $\frac{4}{4}$ is another name for (1).

f) Write the fraction that best describes the uncolored regions of each unit square region above. (a. $\frac{3}{4}$ c. $\frac{1}{4}$ b. $\frac{2}{4}$ d. $\frac{0}{4}$)

2.



Points B and C separate the unit line segment AD into 3 congruent segments.

a) $m\,\overline{AB} = (\frac{1}{3})$

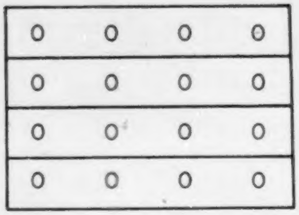
b) $m\,\overline{AC} = (\frac{2}{3})$

c) $m\,\overline{AD} = (\frac{3}{3})$

d) $\frac{3}{3} =$ (1)

Exercise Set 11

1. Look at the picture of a set of objects below.



It has been partitioned into 4 subsets.

The same number of objects is in each subset.

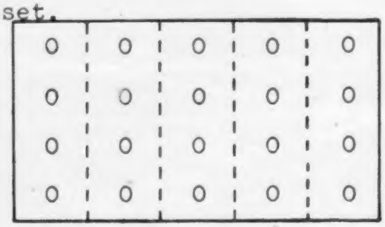
What is $\frac{1}{4}$ of 16? (4)

What is $\frac{2}{4}$ of 16? (8)

What is $\frac{3}{4}$ of 16? (12)

What is $\frac{4}{4}$ of 16? (16)

2. Here is another picture of a set of objects. It has been partitioned into five subsets. The same number of objects is in each subset.



What is $\frac{1}{5}$ of 20? (4)

What is $\frac{2}{5}$ of 20? (8)

What is $\frac{3}{5}$ of 20? (12)

What is $\frac{4}{5}$ of 20? (16)

What is $\frac{5}{5}$ of 20? (20)

3. Complete the following. Use sets of objects if you need them.

a) $\frac{1}{3}$ of 6 is (2)

b) $\frac{1}{2}$ of 4 is (2)

c) $\frac{1}{4}$ of 8 is (2)

d) $\frac{1}{2}$ of 10 is (5)

e) $\frac{2}{3}$ of 6 is (4)

f) $\frac{3}{4}$ of 8 is (6)

g) $\frac{2}{3}$ of 9 is (6)

h) $\frac{2}{5}$ of 10 is (4)

4. Jane bought six doughnuts. She ate $\frac{1}{3}$ of them. How many doughnuts did Jane eat? How many doughnuts did Jane have left? (Jane ate 2 doughnuts and had 4 doughnuts left.)

5. Bill had twenty marbles. He lost $\frac{1}{4}$ of them. How many marbles did Bill lose? How many did he have left?
Bill lost 5 marbles and had 15 marbles left.)

6. Alice had 36 jacks. She traded $\frac{1}{4}$ of them to Mary. How many jacks did Alice trade? How many jacks did Alice have left? Alice traded 9 jacks to Mary and had 27 left.)

7. On the way from the store, Bob dropped a dozen eggs. He looked inside the carton. He found $\frac{3}{4}$ of the eggs broken. How many eggs are there in a dozen? (12) How many eggs were broken? How many eggs were not broken? (There were 9 eggs broken and 3 eggs not broken.)

BRAINTWISTER

John gave Bill sixteen jelly beans. This was $\frac{1}{2}$ of the number John had. How many did John have at the beginning?
(John had 32 jelly beans at the beginning.)

Exercise Set 12

1. There were 20 problems on an arithmetic test. John worked all but $\frac{1}{4}$ of them. How many problems did John finish? (15)

2. $\frac{1}{3}$ of a string of 12 Christmas tree lights had burned out. How many lights had to be replaced? (4)

3. At a sale, books that had been 50¢ were selling for $\frac{1}{2}$ of the regular price. What was the sale price? (25¢)

4. A box which had contained 24 candy bars was two-thirds full. How many candy bars were in the box? (16)

5. A football game is played in 4 quarters. It takes 1 hour of actual playing time to play a game. How many minutes of actual playing time are gone at the end of the third quarter? (45 minutes)

6. There were 6 boys and 3 girls on a softball team. What part of the team was boys? ($\frac{6}{9}$ or $\frac{2}{3}$)

7. The year is separated into four seasons of equal length. What part of the year is each season? ($\frac{1}{4}$)

8. Mary has a collection of 15 dolls. $\frac{2}{3}$ of them represent children from other countries. How many of the dolls represent children from other countries? (10)

9. Jim was making a model of a plane. He needed a single piece of wood $\frac{3}{4}$ of a foot long. He had a piece of wood 8 inches long. Could he use this piece? (No) Why? (He needed $\frac{3}{4}$ of a foot or 9 inches.)

Practice Exercises

1. Place parentheses correctly to make each of the following a true statement. Example a is shown.

a) $(6 + 4) \times 3 = 30$

b) $(8 \times 3) + 5 < 64$

c) $(6 + 3) \times 6 > 24$

d) $2 \times (5 + 4) = 18$

e) $4 \times (16 + 4) > 68$

f) $9 + (6 \div 3) = 11$

g) $8 \times (5 + 3) \not> 43$

h) $(6 \div 3) \times 4 = 8$

i) $(18 \div 6) + 3 \not> 2$

j) $(14 \div 7) + 7 > 1$

k) $(5 \times 8) - 2 > 30$

l) $(25 \div 5) + 8 = 13$

m) $19 + (8 \div 2) > 14$

n) $45 \div (5 + 4) < 13$

o) $(27 + 3) \div 6 = 5$

p) $28 - (7 \times 3) < 63$

q) $(46 + 8) \div 9 = 6$

r) $28 + (21 \div 7) \not> 7$

s) $17 - (4 \times 3) < 39$

t) $(49 \div 7) + 6 = 13$

2. Mixed Addition and Subtraction

a) $327 + 54 = 381$

b) $457 + 218 = 675$

c) $384 + 291 = 675$

d) $384 - 156 = 228$

e) $995 - 768 = 227$

f) $870 - 418 = 452$

g) $2384 - 1963 = 421$

h) $1066 - 883 = 183$

i) $984 + 168 = 1152$

j) $700 - 362 = 338$

k) $1478 + 2388 = 3866$

l) $400 + 583 + 324 = 1307$

m) $1637 - 537 = 1100$

n) $709 - 368 = 341$

o) $37 + 31 + 36 = 104$

p) $801 - 513 = 288$

q) $745 - 508 = 237$

r) $678 + 254 = 932$

s) $2900 - 1256 = 1644$

t) $598 + 303 + 81 = 982$

3. Write the number n represents.

a) 29 + 56 + 37 = n n = 122
b) 700 - 347 = n n = 353
c) 43 × 6 = n n = 258
d) 587 - n = 369 n = 218
e) 77 + 94 + n = 237 n = 66
f) n × 6 = 3708 n = 618
g) 48 + n + 79 = 234 n = 107
h) n = 127 × 5 n = 635
i) 746 - n = 413 n = 333
j) 624 + n = 1141 n = 517
k) n = 737 × 8 n = 5896
l) n + 304 + 488 = 1640 n = 848
m) 4767 = n × 7 n = 681
n) 719 - n = 285 n = 434
o) 8789 + n = 12497 n = 3708
p) 707 × 6 = n n = 4242
q) 8789 - n = 5081 n = 3708
r) 489 + 403 + 950 = n n = 1842
s) n × 9 = 7857 n = 873
t) n - 658 = 758 n = 1416

4. Addition, Subtraction, Multiplication, and Division

a) 1414 - 671 = 743
b) 2157 + 879 = 3036
c) 148 ÷ 4 = 37
d) 367 × 6 = 2202
e) 459 ÷ 9 = 51
f) 309 + 487 + 648 = 1444
g) 475 - 367 = 108
h) 280 ÷ 7 = 40
i) 396 × 7 = 2272
j) 1209 - 688 = 521
k) 278 + 32 + 49 = 359
l) 378 ÷ 6 = 63
m) 439 × 5 = 2195
n) 679 - 327 = 352
o) 136 ÷ 4 = 34
p) 810 + 652 + 934 = 2396
q) 333 × 7 = 2331
r) 652 - 584 = 68
s) 444 ÷ 6 = 74
t) 876 × 4 = 3504

5. **Write the number n represents.**

a) $\frac{1}{2}$ of $12 = n$ $n = 6$

b) $\frac{1}{3}$ of $15 = n$ $n = 5$

c) $\frac{1}{4}$ of $8 = n$ $n = 2$

d) $\frac{1}{9}$ of $9 = n$ $n = 1$

e) $\frac{1}{5}$ of $20 = n$ $n = 4$

f) $\frac{1}{6}$ of $18 = n$ $n = 3$

g) $\frac{2}{3}$ of $9 = n$ $n = 6$

h) $\frac{3}{4}$ of $8 = n$ $n = 6$

i) $\frac{2}{5}$ of $10 = n$ $n = 4$

j) $\frac{1}{8}$ of $16 = n$ $n = 2$

k) $\frac{4}{5}$ of $15 = n$ $n = 12$

l) $\frac{3}{8}$ of $16 = n$ $n = 6$

6. **Find the unknown addend by regrouping.**

Example:

$$\begin{array}{rcl} 462 & = 400 + 60 + 2 = & 400 + 50 + 12 \\ \underline{-157} & = 100 + 50 + 7 = & \underline{100 + 50 + \ \ 7} \\ & & 300 + 00 + \ \ 5 \quad = \quad 305 \end{array}$$

a) $$\begin{array}{r} 609 \\ \underline{-362} \\ 247 \end{array}$$

b) $$\begin{array}{r} 633 \\ \underline{-563} \\ 70 \end{array}$$

c) $$\begin{array}{r} 386 \\ \underline{-219} \\ 167 \end{array}$$

d) $$\begin{array}{r} 890 \\ \underline{-437} \\ 453 \end{array}$$

e) $$\begin{array}{r} 638 \\ \underline{-394} \\ 244 \end{array}$$

f) $$\begin{array}{r} 853 \\ \underline{-628} \\ 225 \end{array}$$

g) $$\begin{array}{r} 493 \\ \underline{-316} \\ 177 \end{array}$$

h) $$\begin{array}{r} 761 \\ \underline{-257} \\ 504 \end{array}$$

i) $$\begin{array}{r} 780 \\ \underline{-333} \\ 447 \end{array}$$

j) $$\begin{array}{r} 956 \\ \underline{-247} \\ 709 \end{array}$$

7. Multiplication and Division

a) 72 ÷ 8 = 9

b) 789 × 5 = 3945

c) 725 ÷ 5 = 145

d) 864 × 6 = 5184

e) 408 ÷ 4 = 102

f) 904 × 7 = 6328

g) 824 ÷ 4 = 206

h) 496 × 6 = 2976

i) 654 ÷ 3 = 218

j) 730 × 9 = 6570

k) 378 ÷ 3 = 126

l) 257 × 4 = 1028

m) 2488 ÷ 8 = 311

n) 319 × 8 = 2552

o) 580 ÷ 5 = 116

p) 509 × 7 = 5663

q) 789 ÷ 3 = 263

r) 156 × 9 = 1404

s) 217 ÷ 7 = 31

t) 697 × 3 = 2091

8. Write the numeral for each blank that makes a true sentence.

Example a is shown.

a) (4 ÷ 2) × 9 = 18

b) (7 × 9) - 29 = 34

c) (8 × 8) - 14 = 50

d) (12 × 6) - 65 = 7

e) (6 × 8) - 39 = 9

f) (4 × 6) + 6 = 30

g) (6 × 6) - 22 = 14

h) (54 - 47 × 7 = 49

i) (11 - 5) × 9 = 54

j) (8 + 20) - 14 = 14

k) (9 × 4) + 8 = 44

l) (6 × 6) - 15 = 21

m) (7 × 8) + 16 = 72

n) (8 × 5) - 12 = 28

o) (7 × 9)+ 12 = 75

p) (5 × 7) - 14 = 21

q) (6 × 9) - 6 = 48

r) (7 × 6) + 13 = 55

s) (6 × 6) - 7 = 29

t) (7 × 9) + 9 = 72

Review

SET I

Part A

1. Using the symbol >, = , or < make each of the following a true sentence.

a) $\frac{2}{3}$ foot $\underline{\quad < \quad}$ 12 inches

b) 24 inches $\underline{\quad > \quad}$ $\frac{1}{2}$ yard

c) 1 pint $\underline{\quad = \quad}$ $\frac{1}{2}$ quart

d) $\frac{1}{2}$ hour $\underline{\quad = \quad}$ 30 minutes

e) $\frac{2}{3}$ yard $\underline{\quad = \quad}$ 2 feet

f) 12 inches $\underline{\quad > \quad}$ $\frac{1}{4}$ yard

g) $\frac{1}{2}$ quart $\underline{\quad < \quad}$ 2 pints

h) 15 minutes $\underline{\quad = \quad}$ $\frac{1}{4}$ hour

i) 4 feet $\underline{\quad < \quad}$ $1\frac{1}{2}$ yards

f) 9 inches $\underline{\quad < \quad}$ $\frac{1}{3}$ yard

2. Arrange in order of size from smallest to largest.

a) $\frac{2}{3}, \frac{1}{3}, \frac{6}{3}, \frac{4}{3}, \frac{8}{3}$ ($\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{6}{3}, \frac{8}{3}$)

b) $\frac{1}{2}, \frac{1}{3}, \frac{1}{7}, \frac{1}{5}, \frac{1}{9}$ ($\frac{1}{9}, \frac{1}{7}, \frac{1}{5}, \frac{1}{3}, \frac{1}{2}$)

c) $\frac{3}{6}, \frac{3}{3}, \frac{3}{4}, \frac{3}{8}, \frac{3}{12}$ ($\frac{3}{12}, \frac{3}{8}, \frac{3}{6}, \frac{3}{4}, \frac{3}{3}$)

d) $\frac{1}{2}, \frac{3}{4}, \frac{2}{8}, \frac{4}{4}, \frac{1}{8}$ ($\frac{1}{8}, \frac{2}{8}, \frac{1}{2}, \frac{3}{4}, \frac{4}{4}$)

e) $\frac{4}{5}, \frac{3}{5}, \frac{1}{5}, \frac{2}{5}, \frac{7}{5}$ ($\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{7}{5}$)

f) 1 ft., 6 in., $\frac{3}{4}$ ft., $\frac{1}{4}$ ft. ($\frac{1}{4}$ ft, 6 in., $\frac{3}{4}$ ft., 1 ft)

g) 45 min., $\frac{1}{2}$ hr., 60 min., $\frac{1}{4}$ hr. ($\frac{1}{4}$ hr, $\frac{1}{2}$ hr., 45 min, 60 min.)

h) 1 ton, 1 lb., $\frac{1}{2}$ lb., 1 oz. (1 oz., $\frac{1}{2}$ lb., 1 lb., 1 ton)

i) 1 pt., 1 qt., 1 cup, 1 gal., (1 cup, 1 pt., 1 qt., 1 gal)

j) $\frac{1}{3}$ in., $\frac{2}{3}$ in., $\frac{5}{6}$ in., $\frac{1}{6}$ in. ($\frac{1}{6}$ in., $\frac{1}{3}$ in., $\frac{2}{3}$ in., $\frac{5}{6}$ in.)

3. N = {The named points A, B}

What is the greatest number of line segments that can have endpoints in Set N ? (One)

Name the segment(s). ($\overline{AB}$)

4. M = {The named points C, D, E}

What is the greatest number of line segments that can have endpoints in Set M ? (Three)

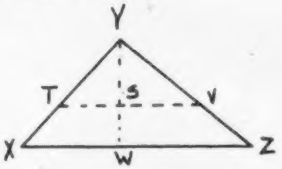
Name the segment(s). ($\overline{CD}$, $\overline{DE}$, $\overline{EC}$)

5. R = {The named points G, F, H, I}

What is the greatest number of line segments that can have endpoints in Set R ? (Six)

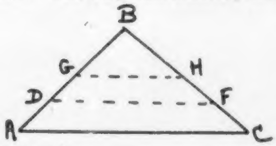
Name the segment(s). ($\overline{GF}$, $\overline{GH}$, $\overline{GI}$, $\overline{HI}$, $\overline{HF}$, $\overline{FI}$)

6. Draw two line segments to make three quadrilaterals and five new triangles out of △ XYZ



(△ TVY, △ TSY, △ VSY, △ XYW, △ ZYW, ▭ XTVZ, ▭ XTSW, ▭ WSVZ)

7. Draw two line segments to make two new triangles and three quadrilaterals out of △ ABC.



(△ DBF, △ BGH, ▭ DFCA, ▭ AGHC, ▭ DGHF)

8. $4 < 5 < 8$ means 4 is less than 5 and 5 is less than 8.

Write these sentences the shorter way.

a) $32 < 34$ and $34 < 40$ ($32 < 34 < 40$) (Answers will vary)

b) $\frac{1}{4} < \frac{2}{4}$ and $\frac{2}{4} < \frac{3}{4}$ ($\frac{1}{4} < \frac{2}{4} < \frac{3}{4}$)

c) $112 < 115$ and $115 < 117$ ($112 < 115 < 117$)

d) $\frac{1}{2} < \frac{3}{4}$ and $\frac{3}{4} < \frac{7}{8}$ ($\frac{1}{2} < \frac{3}{4} < \frac{7}{8}$)

9. Find the number that n represents in the following.

a) $14 < n < 16$ $(n = 15)$ d) $\frac{3}{8} < n < \frac{5}{8}$ $(n = \frac{4}{8}$ or $\frac{1}{2})$

b) $\frac{1}{3} < n < \frac{3}{3}$ $(n = \frac{2}{3})$ e) $9 < n < \frac{20}{2}$ $(n = \frac{19}{2}$ or $9\frac{1}{2})$

c) $786 < n < 788$ $(n = 787)$ f) $\frac{1}{2} < n < 1\frac{1}{2}$ $(n = \frac{2}{2}$ or $1)$

(Answers will vary)

10. What is it?

a) A model that has 3 rectangular regions and 2 triangular regions for faces. (triangular prism)

b) A model that has four triangular regions for faces.
(pyramid)

c) A model that has six rectangular regions for faces.
(rectangular prism)

d) A model that has one rectangular region and two circular regions for faces. (cylinder)

e) A model whose edges form right angles only.
(rectangular prism)

f) A model that has a circular region and a half circular region for faces. (cone)

11. These statements are comparing the length of line segments. Complete these to make them true statements. Examples a and b are done for you.

a) 4 ft. is $\underline{\ 2\ }$ times as long as 2 ft.

b) 2 ft. is $\underline{\ \frac{1}{2}\ }$ as long as 4 ft.

c) 9 in. is $\underline{\ 3\ }$ times as long as 3 in..

d) 3 in. is $\underline{\ \frac{1}{3}\ }$ as long as 9 in..

e) $\underline{12}$ yd is 2 times as long as 6 yd..

f. 6 yd. is $\underline{\ \frac{1}{2}\ }$ as long as 12 yd.

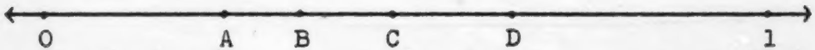
g. 15 min. is $\underline{\ \frac{1}{3}\ }$ as long as 45 min..

h. 45 min. is $\underline{\ 3\ }$ times as long as 15 min..

12. Match the Standard Unit of Measure from Column I with the Item you would use it to measure from Column II.

Column I		Column II
a) cup	j	Length of your pencil
b) feet	f	Bottle of milk
c) seconds	i	Distance between cities
d) hours	g	Length of a football field
e) days	c	Timing a running race
f) quart	h	Time at recess
g) yards	e	Time until a birthday
h) minutes	d	Time for sleeping
i) miles	b	Height of a tree
j) inches	a	Sugar for a recipe

13. Write 4 different fraction names for each of the points labeled on this number line.



(A $\frac{1}{4}, \frac{2}{8}, \frac{4}{16}, \frac{8}{32}$; B $\frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}$; C $\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}$
D $\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}$) (Answers will vary)

14. Find the perimeter of the following:

a) A polygon with sides whose measures are 16, 28, and 32 in inches.(Perimeter 76 in.)

b) An equilateral triangle with the measure of one of its sides 14 in feet.(Perimeter 42 ft.)

c) A polygon with 6 congruent sides, the measure of one side is 35 in centimeters.(perimeter 210 centimeters)

d) A square, one side of which has the measure of 7 in meters. (Perimeter 28 meters)

e) A polygon with 2 sides whose measures have the sum of 8 yards and 3 sides whose measures have the sum of 15 in yards.(Perimeter 23 yards)

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Part B

Write a mathematical sentence (or two sentences if necessary) for each problem and solve. Write an answer sentence.

1. The Davis family uses 8 eggs for breakfast. What part of a dozen eggs is left? ($12 - 8 = t$, $t = 4$, $4 \div 12 = n$, $n = \frac{4}{12}$ or $\frac{1}{3}$. They have $\frac{4}{12}$ of a dozen eggs left.)
2. Eddie earns 75 cents on Monday, 50 cents on Wednesday, and 75 cents on Friday mowing lawns. How much will he earn in six weeks? $(75 + 50 + 75) \times 6 = n$ or $75 + 50 + 75 = c$
$c = 200$, $200 \times 6 = n$ $n = 1200$ Eddie will earn $12.00)
3. The school bus runs 7 miles on a gallon of gasoline. Each week the bus goes an average of 882 miles. How much gasoline will the bus use in four weeks? $(882 \div 7) \times 4 = n$, or
$882 \div 7 = g$, $g = 126 \times 4 = n$, $n = 504$ The bus will use 504 gallons of gasoline.)
4. Wendy called the feed store to order feed for a month for her horse. She bought 5 bales of hay at $1.75 a bale and 100 lb. of oats at $5.30. How much will this month's feed bill be? $(5 \times 175) + 530 = n$ or $5 \times 175 = h$, $h + 530 = n$
$n = 1405$. This month's feed bill will be $14.05
5. Tom hit a softball 135 ft. Randy hit the ball 25 ft. farther than Tom. How far did Randy hit the ball? ($135 + 25 = n$
$n = 160$ Randy hit the ball 160 feet.
6. It is 347 air miles from San Francisco to Los Angeles, 1240 air miles on to Dallas, 443 air miles from Dallas to New Orleans, then 669 air miles on to Miami. How many air miles is it, by this route, from San Francisco to Miami?
($347 + 1240 + 443 + 669 = n$, $n = 2699$ It is 2699 air miles from San Francisco to Miami.

7. The nurse found Janice to be 4 ft. 4 in tall, Linda 4 ft. 11 in. tall, and Maria 4 ft. 9 in tall. How much taller than Janice is Linda? (4 ft. 11 in - 4 ft. 4 in = n
n = 7 in. Linda is 7 inches taller than Janice.)

8. In the standing board-jump Pat's best jump was 5 ft. 3 in. while Roy's best jump was 6 ft. 2 in. Roy's jump was how much better than Pat's? (6 ft. 2 in. - 5 ft. 3 in = n
n = 11 in. Roy's jump was 11 inches better than Pat's.)

9. For his birthday Tom received a new baseball bat that is 24 inches long. The bat's length is what part of a yard?
($\frac{24}{36} = n$ $n = \frac{2}{3}$ The bat is $\frac{2}{3}$ yard long.)

10. Joe delivers 56 papers each day. How many papers does he deliver in 28 days? (56 × 28 = t, t = 1,568 Joe delivers 1,568 papers in 28 days.)

11. Susan buys 2 dozen cookies for 30 cents a dozen and a cake for 80 cents. How much does she pay the clerk?
(30 × 2 = a or 30 + 30 = a a = 60 60 + 80 = c c = 140
Susan pays the clerk $1.40.)

Braintwisters

1. You have a 30 inch board that you have to cut in 5 pieces, each 6 inches long. It takes five minutes to make each cut. How many minutes will it take you to cut the five pieces? (4 × 5 = 20)

2. An inchworm was climbing a tree 5 ft high. He climbed three inches every day and slipped back two inches every night. How many days will it take him to reach the top?
(5 × 12) - 2 = n n = 58 58 days)

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